

Bootstrap percolation with inhibition

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Abstract

Bootstrap percolation is a prominent framework for studying the spreading of activity on a graph. We begin with an initial set of active vertices. The process then proceeds in rounds, and further vertices become active as soon as they have a certain number of active neighbors. A recurring feature in bootstrap percolation theory is an ‘all-or-nothing’ phenomenon: either the size of the starting set is so small that the process stops very soon, or it percolates (almost) completely.

Motivated by several important phenomena observed in various types of real-world networks we propose in this work a variant of bootstrap percolation that exhibits a vastly different behavior. Our graphs have two types of vertices: some of them obstruct the diffusion, while the others facilitate it. We study the effect of this setting by analyzing the process on Erdős-Rényi random graphs. Our main findings are two-fold. First we show that the presence of vertices hindering the diffusion does not result in a stable behavior: tiny changes in the size of the starting set can dramatically influence the size of the final active set. In particular, the process is non-monotone: a larger starting set can result in a smaller final set. In the second part of the paper we show that this phenomenon arises from the round-based approach: if we move to a continuous time model in which every edge draws its transmission time randomly, then we gain stability, and the process stops with an active set that contains a non-trivial constant fraction of all vertices. Moreover, we show that in the continuous time model percolation occurs significantly faster compared to the classical round-based model. Our findings are in line with empirical observations and demonstrate the importance of introducing various types of vertex behaviors in the mathematical model.

1 Introduction

Bootstrap percolation is a classical and well-studied mathematical framework for the spreading of activity on a given graph. One starts with an initial set of *active* vertices; this set may be

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chosen randomly or deterministically. The process then proceeds in rounds, and further vertices become active as soon as they have at least k active neighbors, where $k \in \mathbb{N}$ is a parameter of the process. The process is said to *percolate* if all vertices eventually turn active.

This process was first studied in 1979 on Bethe lattices by Chalupa, Leath and Reich [7] to model demagnetisation in magnetic crystals. If we choose the starting set randomly, then one would expect that for many graphs there is a *percolation threshold* such that if the number of starting vertices is above this threshold, then the process percolates, whereas if it is below the threshold, it does not. For example, such a threshold has been determined for finite grids by Balogh, Bollobás, Duminil-Copin and Morris [3] and for the Erdős-Rényi random graph by Janson, Łuczak, Turova, and Vallier [12]. The problem has also been studied on various other graphs [1, 5, 4] and models, as for example in cellular automata [16, 10]. In all of these examples we observe an “all-or-nothing” phenomenon: if the size of the starting set is too small, then the process stops rather quickly, and otherwise, it spreads to a level that includes (almost) all vertices of the underlying graph. In a way, this is not too surprising, as the likelihood that a vertex has k active neighbors increases with the total number of active vertices.

The aim of this paper is to study percolation processes with inhibition that can restrict further dissemination of activity. As an example, consider the diffusion of an innovation in a society. When a new product is introduced to the market people may like and promote it or they may dislike and denigrate it. If we now assume that people buy the product as soon as they get, say, k more positive feedbacks than negative ones from their neighbors, we have a bootstrap percolation process with inhibition. Another example is a phenomenon called *input normalization* in neuroscience, cf. [6] for a review. This refers to the following well-studied observation: when a signal activates a small part of a local ensemble of neurons, the activity spreads through to recurrent connections. But only up to a certain point. Then inhibitory neurons are strong enough to stop a further spread of activation. In this way, very different input strengths can lead to similar levels of activity that never surpass a certain upper bound. Such an effect has been observed experimentally in many species [9, 15, 14].

In this paper, we consider the Erdős-Rényi random graph $G_{n,p}$ with two types of vertices: *inhibitory* vertices (those obstructing the diffusion) and *excitatory* vertices (those facilitating the diffusion). First we show that in the standard, round-based percolation model the introduction of inhibitory vertices does not result in a stable behavior: either inhibition has essentially no effect, or tiny changes in the size of the starting set can dramatically influence the size of the final active set. In particular, the process is non-monotone: a larger starting set can result in a smaller final set. In the second part of the paper we show that such a phenomenon is actually the result of the round-based approach: if we move to a continuous time model in which every edge draws its transmission time randomly according to an exponential distribution, then normalization is an automatic and intrinsic property of the process. Moreover, we find that random edge delays accelerate percolation dramatically: for transmission delays that are distributed according to independent exponential distributions with mean one the time to activate all vertices reduces from $\Theta(\log \log n)$ in the round based model to $O(1)$ time in the asynchronous model.

1.1 Model and results

The classical bootstrap percolation process on Erdős-Rényi random graphs was studied by Janson, Łuczak, Turova, and Vallier [12]. This process starts with a random active subset of size a of the vertices. The process then proceeds in rounds, where in each round all non-active vertices that have at least k active neighbors also become active, and remain so forever. A percolation process *percolates completely* if there is some round in which every vertex is active. It *almost*

percolates if there is some round in which $n - o(n)$ vertices are active. Janson et al. showed that for $1/n \ll p \ll n^{-1/k}$ there exists a threshold

$$\mathbf{a}_{\text{th}}(n, p, k) = (1 - 1/k) \left(\frac{(k-1)!}{np^k} \right)^{1/(k-1)}$$

such that for every $\varepsilon > 0$, a.a.s.¹ the process almost percolates for $\mathbf{a} > (1 + \varepsilon)\mathbf{a}_{\text{th}}$, and a.a.s. it stays forever at $O(\mathbf{a}_{\text{th}})$ active vertices if $\mathbf{a} < (1 - \varepsilon)\mathbf{a}_{\text{th}}$. They also showed that for starting sets of size $\mathbf{a} = (1 + \varepsilon)\mathbf{a}_{\text{th}}$ the process almost percolates in $\log_k \log(pn) + O(1)$ rounds, where the hidden constant depends only on ε . Observe that this result immediately carries over to *directed* random graphs (in which activation requires k active in-neighbours) if we insert each directed edge with probability p .

We extend the standard model by allowing inhibitory vertices: we assume that each of the n vertices is *inhibitory* with probability τ and *excitatory* with probability $1 - \tau$, independently. To be slightly more general, we also introduce an additional parameter $0 < \gamma \leq 1/p$ and insert each directed edge with excitatory (inhibitory) origin independently with probability p (with probability γp). The process is similar to the classical bootstrap percolation with one crucial difference: a previously inactive vertex v turns active in some round i if after round $i - 1$ the number of active excitatory neighbors of v *exceeds* the number of its active inhibitory neighbors by at least k . We generalize the threshold function \mathbf{a}_{th} so that it now also depends on τ :

$$\mathbf{a}_{\text{th}} = \mathbf{a}_{\text{th}}(n, p, k, \tau) = (1 - 1/k) \left(\frac{(k-1)!}{(1 - \tau)^k np^k} \right)^{1/(k-1)}. \quad (1)$$

Note that the threshold does not depend on the inhibition excess γ . Note further that, compared to the threshold $\mathbf{a}_{\text{th}}(n, p, k, 0)$ for the case without inhibition, there is an additional factor of $(1 - \tau)^k$ in the denominator. This factor can be interpreted in the following way: clearly, a necessary condition for percolation is that the process percolates in the subgraph induced by the excitatory vertices, which has $(1 + o(1))(1 - \tau)n$ vertices a.a.s.. If we choose a random starting set of size \mathbf{a} , then this starting set will contain $(1 + o(1))(1 - \tau)\mathbf{a}$ excitatory vertices a.a.s.. Then, by the result for the process without inhibition, the process will not percolate if

$$(1 - \tau)\mathbf{a} \leq (1 - \varepsilon)\mathbf{a}_{\text{th}}((1 - \tau)n, p, k, 0),$$

or, equivalently, if $\mathbf{a} \leq (1 - \varepsilon)\mathbf{a}_{\text{th}}(n, p, k, \tau)$. In particular, we can restrict our analysis to the case $\mathbf{a} \geq \mathbf{a}_{\text{th}}$, since, by the results of Janson, Łuczak, Turova and Vallier [12], the process with $\mathbf{a} \leq (1 - \varepsilon)\mathbf{a}_{\text{th}}$ will stop with $O(\mathbf{a})$ active vertices.

Our results for this process are collected in the following theorem.

Theorem 1. *Let $\varepsilon, \tau, \gamma > 0$, $k \geq 2$ and assume $1/n \ll p \ll n^{-1/k}$ and $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}(n, p, k, \tau)$. Then the bootstrap percolation process with inhibition a.a.s. satisfies the following.*

- (i) *For $\tau < 1/(1 + \gamma)$ the process almost percolates in $\log_k \log_{(\mathbf{a}/\mathbf{a}_{\text{th}})}(np) + O(1)$ rounds (as it does in the case without inhibition). If, additionally, $p = \omega(\log n/n)$, then the process percolates completely in the same number of rounds.*
- (ii) *For $\tau > 1/(1 + \gamma)$ and $\mathbf{a} \geq (\log n)^{2+\varepsilon}$ and $p = \omega(\log n/n)$ the process is chaotic in the following sense: for every constant $C_1 > 0$ there exists a constant $C_2 > C_1$ such that for every target function f with $(\log n)/p \ll f(n) \ll n$, there exists a function $c: \mathbb{N} \rightarrow [C_1, C_2]$ such that if one starts the process with $\mathbf{a} = \lfloor c(n)\mathbf{a}_{\text{th}} \rfloor$ vertices, then it stops with $\mathbf{a}^* = (1 + o(1))f(n)$ active vertices a.a.s..*

¹Asymptotically almost surely, that is, with probability tending to one as $n \rightarrow \infty$.

In other words, if $\tau < 1/(1 + \gamma)$ then inhibition has basically no effect on the outcome of the process: the process behaves similar as in the classical case with $(1 - \tau)n$ vertices and no inhibition. On the other hand, if $\tau > 1/(1 + \gamma)$, then the outcome of the process depends in a rather unstable way on the size of the initially active set: by changing the size of the starting set by a constant factor, we can change the number of active vertices at the end of the process drastically; in particular, the number of active vertices at the end of the process is non-monotonic in the size of the starting set. We remark that the condition $\mathfrak{a}_{\text{th}} \geq (\log n)^{2+\varepsilon}$ is essentially best possible: for $\mathfrak{a}_{\text{th}} \leq (\log n)^{2-\varepsilon}$ the statement of the theorem is not true, cf. the argument following Theorem 12 on page 11. If one weakens the conditions of (ii) by just requiring $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}$ or $p \gg 1/n$, then it is still true that the process is unstable, but one cannot predict where it ends.

A main feature of the classical bootstrap percolation processes is that activation takes place in rounds. This phenomenon can be interpreted in the following way: when a vertex turns active, then this information needs exactly one time unit to reach its neighbors. In the second part of our paper we drop this assumption and replace this synchronous model with an asynchronous one: each edge independently draws a random transmission delay δ from an exponential distribution with expectation one, and the information that the neighbor is active requires time δ to travel from one vertex to another. The activation rule itself remains unchanged: a vertex turns active as soon as it is aware that k of its neighbors are active (in the process without inhibition), or as soon as it has notice of k more active excitatory than inhibitory neighbors (in the general case). Although the expected transmission delay is one – as it is deterministically in the synchronous model – it turns out that quantitatively and qualitatively the percolation process changes rather dramatically.

Theorem 2. *Let $\varepsilon, \tau, \gamma > 0$, $k \geq 2$ and assume $1/n \ll p \ll n^{-1/k}$ and $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}(n, p, k, \tau)$. Then there exists a constant $T = T(\varepsilon, k) \geq 0$ such that the asynchronous bootstrap percolation process with $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}$ and $n^{-1} \ll p \ll n^{-1/k}$ a.a.s. satisfies the following.*

- (i) *If $\tau < 1/(1 + \gamma)$, then the process almost percolates in time T . If, additionally, $p = \omega(\log n/n)$, then the process percolates completely within time T .*
- (ii) *If $\tau \geq 1/(1 + \gamma)$ and if $\mathfrak{a} = o(n)$, then a.a.s. there are $(1 - \tau)^k n / (\gamma \tau)^k + o(n)$ active vertices at time T . Also, the process stops with $(1 - \tau)^k n / (\gamma \tau)^k + o(n)$ active vertices.*

Note that this theorem implies two interesting phenomena. On the one hand, we see that the asynchronous version accelerates the process dramatically: it essentially stops after *constant time*, as opposed to the roughly $\log_k \log(pn)$ rounds it takes in the synchronous model. On the other hand, we see that the final size of the process only depends on the parameters τ and γ but not on the size of the initial set, in sharp contrast to the synchronous model. Therefore, by choosing the parameters τ and γ appropriately, we can realize a normalization for an arbitrary (linear) target size of the finally activated set.

We note that recently it was shown in [13] that Theorem 2 also implies similar results for other random graph models.

1.2 Outline

In [12] it is shown among other things that the classical bootstrap percolation process on $G_{n,p}$ (without inhibition) consists of three phases. While the active set is still very small (close to the threshold size \mathfrak{a}_{th}), the active set grows only by a small factor in each round. Once the size of the active set is asymptotically larger than \mathfrak{a}_{th} , the growth of the active set picks up momentum, and we call this the *explosion phase*. Finally, once the active set has size at least $1/p$, the process

terminates in at most two more rounds (provided $p \gg \log n/n$): one round to activate a linear subset of the vertices, and a possible second round to activate all remaining vertices. A similar situation occurs also in the process with inhibition. In order to prove Theorem 1 we need to track the size of the active set very precisely during the first two phases (as a function of the size of the starting set). In principle, this seems like an easy task: given an active set of size x , we expect that in the next round we activate $(n-x) \cdot \Pr[\text{Bin}(x, p) \geq k]$ additional vertices. The problem is, of course, that such a simple approach ignores the dependencies between rounds. We overcome this issue by defining a different probability space (Section 2) that describes the same process but is more amenable to a formal analysis. In this section we also prove some general properties of the percolation process that apply both to the synchronous and the asynchronous case. In Section 3 we then use these preliminaries to first describe the evolution of the size of the active set as a function of the number rounds very precisely (Theorem 12) and subsequently use this result to prove Theorem 1. In Section 4 we then consider the asynchronous version of the process. As we will see, this process behaves similarly as the synchronous version in the very early stage of the process (while still close to the threshold) but then speeds up considerably. Theorem 2 then follows from the fact that the sum of the incoming signals (positive minus negative ones) essentially performs a random walk where the bias is a function of τ and γ .

2 Preliminaries and Definitions

The aim of this section is to define a general *bootstrap percolation process* which subsumes both the synchronous and the asynchronous case, and to prove some basic properties of this process.

2.1 Formal definition of the percolation process

In this section we describe a version of the bootstrap percolation process that is particularly amenable to its analysis. We first activate the vertices in the starting set, assuming without loss of generality that this set consists of the vertices $1, \dots, a$. Then for each $s \geq 1$ we provide just enough information with the s -th active vertex to determine whether a $(s+1)$ -st vertex is activated. Crucially, this information does *not* require knowledge of the labels of the active vertices. In this way we can determine properties of active sets of a certain size, without actually knowing which vertices belong to this set. We now turn to the details.

Let $n \in \mathbb{N}$, let $p, \tau \in [0, 1]$, and let $\gamma \in [0, 1/p]$. Let Φ be a random variable taking values in the positive reals. Define the product probability space

$$(\Omega, \mathcal{A}, \Pr) = \prod_{1 \leq i, v \leq n} (X_{iv}^+ \times X_{iv}^- \times \Phi_{iv}) \times \prod_{i=1}^n \Psi_i,$$

where

- X_{iv}^+ is the probability space of a Bernoulli random variable with parameter p ,
- X_{iv}^- is the probability space of a Bernoulli random variable with parameter γp ,
- Ψ_i is the probability space with $\Pr[-1] = \tau$ and $\Pr[+1] = 1 - \tau$, and
- Φ_{iv} is the probability space on $\mathbb{R}_{>0}$ given by the distribution of Φ .

By abuse of notation, we also denote the random variables corresponding to the spaces X_{iv}^+ , X_{iv}^- , Φ_{iv} , and Ψ_i again by X_{iv}^+ , X_{iv}^- , Φ_{iv} , and Ψ_i , respectively. It will always be clear from the context which interpretation we have in mind.

Before precisely defining the percolation process we give the intended interpretations of these random variables. By symmetry, we assume that the initially active set is $[a] = \{1, \dots, a\}$. Define

$$x_i := i \quad \text{for all } 1 \leq i \leq a.$$

In general, x_i will be the label of the i -th vertex that becomes active in the percolation process, where, if several vertices should become active simultaneously ties are broken arbitrarily, for example by the natural ordering of the vertices. Then

- Ψ_i determines the sign of vertex x_i . That is, vertex x_i is inhibitory if and only if $\Psi_i = -1$, which happens with probability τ , and excitatory otherwise;
- X_{iv}^- and X_{iv}^+ describe whether there is a directed edge from vertex x_i to v : there is a directed edge from x_i to v exactly if either $\Psi_i X_{iv}^+ = 1$ or $\Psi_i X_{iv}^- = -1$. Note that the roles of i and v are not interchangeable: while v represents a vertex of the underlying graph, i represents the *index* of the i -th vertex that becomes active.
- Φ_{iv} describes the delay of the edge (x_i, v) . In the synchronous model, the delay is a constant of value 1, while in the asynchronous model, it is an exponentially distributed random variable with parameter 1. Note that for ease of analysis we define these random variables regardless of whether $X_{iv} = 1$ or not.

For every $s \in [n]$, we define random variables $E_s, I_s: \Omega \rightarrow \mathcal{P}([s])$ by

$$E_s := \{i \in [s] \mid \Psi_i = +1\} \quad \text{and} \quad I_s := \{i \in [s] \mid \Psi_i = -1\},$$

respectively. These are the sets containing the indices of the active excitatory resp. inhibitory vertices at the time at which exactly s vertices are active.

We can now describe formally how elements $\omega \in \Omega$ define a *percolation process*

$$((x_1, t_1), \dots, (x_n, t_n))$$

with starting set $[a]$ on $G_{n,p}$. First, activate all the vertices in $[a]$ at time $t = 0$ by letting $x_s = s$ and $t_s = 0$ for all $1 \leq s \leq a$. Assume now that active vertices x_1, \dots, x_s are given, where $s \geq a$. Also assume that for each such vertex x_i we know the time t_i when it turned active.

Then x_{s+1} is defined as follows. First, for each vertex $v \in [n] \setminus \{x_1, \dots, x_s\}$ we determine the earliest time $t_v^{(s)}$ at which v has received k more excitatory than inhibitory signals from the set $\{x_1, \dots, x_s\}$:

$$t_v^{(s)} := \min \left\{ t \in \mathbb{R}_{\geq 0} \mid |\{i \in E_s \mid X_{iv} = 1 \text{ and } t_i + \Phi_{iv} \leq t\}| \geq k + |\{i \in I_s \mid X_{iv} = 1 \text{ and } t_i + \Phi_{iv} \leq t\}| \right\},$$

where $\min \emptyset = \infty$. If there is some vertex v for which $t_v^{(s)} < \infty$, then we define

$$t_{s+1} := \min \{t_v^{(s)} \mid v \in [n] \setminus \{x_1, \dots, x_s\}\}.$$

In this case, let $J := \{v \in [n] \setminus \{x_1, \dots, x_s\} \mid t_v^{(s)} = t_{s+1}\}$ and $j := |J|$. We set $t_{s+i} := t_{s+1}$ for $2 \leq i \leq j$, and we let $x_{s+1} < \dots < x_{s+j}$ be the (uniquely determined) vertices such that

$$\{x_{s+1}, \dots, x_{s+j}\} = J.$$

If, on the other hand, we have $t_v^{(s)} = \infty$ for all $v \in [n] \setminus \{x_1, \dots, x_s\}$, then the process stops and we set $t_{s'} := \infty$ and $x_{s'} := \min \{v \in [n] \setminus \{x_1, \dots, x_{s'-1}\}\}$ for all $s' \geq s+1$, i.e., we enumerate all remaining vertices by increasing label.

Finally, we introduce some more useful notation. For every $s \in [n]$ and $v \in [n]$, we define the random variables

$$N_s^+(v) := \sum_{i \in E_s} X_{iv}^+ \quad \text{and} \quad N_s^-(v) := \sum_{i \in I_s} X_{iv}^-.$$

Note that $N_s^+(v)$ and $N_s^-(v)$ are the number of excitatory and inhibitory active neighbors of v at the time at which exactly s vertices are active. For brevity, we also use $N_s(v) := N_s^+(v) + N_s^-(v)$.

Remark 3. *From the definition of the probability space it follows immediately that for all positive integers e , the conditional distributions of $N_s^+(v)$ and $N_s^-(v)$ given $|E_s| = e$ are binomial. More specifically, for every $0 \leq x \leq s$, we have*

$$\Pr[N_s^+(v) = x \mid |E_s| = e] = \binom{e}{x} p^x (1-p)^{e-x}$$

and

$$\Pr[N_s^-(v) = x \mid |E_s| = e] = \binom{s-e}{x} (\gamma p)^x (1-\gamma p)^{s-e-x}.$$

Also, for distinct vertices v and w , the random variables $N_s^+(v)$, $N_s^-(v)$, $N_s^+(w)$ and $N_s^-(w)$ are mutually conditionally independent, given the value of $|E_s|$. In addition, note that $N_s(v) \sim \text{Bin}(s, \hat{p})$, for every $s, v \in [v]$, where $\hat{p} = (1-\tau)p + \tau\gamma p$.

We will make frequent use of the following concentration bounds on the binomial distribution [11].

Lemma 4 (Chernoff). *Let X_1, \dots, X_n be independent Bernoulli variables with $\Pr[X_i = 1] = p$ and $\Pr[X_i = 0] = 1 - p$ for all $1 \leq i \leq n$, and let $X = \sum_{i=1}^n X_i$. Then for every $0 \leq \delta \leq 1$,*

$$\Pr[X \geq (1+\delta)np] \leq e^{-\delta^2 np/3} \quad \text{and} \quad \Pr[X \leq (1-\delta)np] \leq e^{-\delta^2 np/3}.$$

2.2 General properties of the percolation process

In this subsection we prove some properties of the probability space that are independent of the distribution of the transmission delays Φ_{iv} . These results thus apply equally in the synchronous and the asynchronous case.

Let us start with the following simple fact, which states that at every point in time, the numbers of active excitatory and inhibitory vertices are close to their expectations.

Lemma 5. *Let $\delta_0 = \delta_0(n) \in (0, 1/2)$ be such that $\delta_0^2 \mathbf{a} = \omega(-\log \delta_0)$. Then a.a.s. the percolation process satisfies*

$$|E_s| \in (1 \pm \delta_0)(1-\tau)s \quad \text{and} \quad |I_s| \in (1 \pm \delta_0)\tau s$$

for all $s \geq \mathbf{a}$.

Remark 6. We will apply this lemma in two settings: first, when δ_0 is constant and $\mathfrak{a} = \omega(1)$, second, when $\delta_0 = (\log n)^{-1-\varepsilon/3}/(10k)$ and $\mathfrak{a} \geq (\log n)^{2+\varepsilon}$, for some constant $\varepsilon > 0$. Note that in both cases, the condition $\delta_0^2 \mathfrak{a} = \omega(-\log \delta_0)$ is satisfied.

Proof of Lemma 5. If $\tau = 0$ or $\tau = 1$ there is nothing to show. So assume $0 < \tau < 1$. It follows directly from the definitions that for every $s \in [n]$, we have $|E_s| \sim \text{Bin}(s, 1 - \tau)$. Then Lemma 4 and the union bound imply that

$$\Pr[\exists s \geq \mathfrak{a} : |E_s| \notin (1 \pm \delta_0)(1 - \tau)s] \leq \sum_{s \geq \mathfrak{a}} 2e^{-\frac{\delta_0^2}{3}(1-\tau)s} = o(1),$$

where we used that $\delta_0^2 \mathfrak{a} = \omega(-\log \delta_0)$ and routine calculations to obtain the last equality. The statement for $|I_s|$ is proved similarly. \square

Our primary goal in this subsection is to introduce a general method to prove that the process reaches a certain number of active vertices in a certain period of time. To do this, for every $s \in [n]$ and $r \in \mathbb{R}_{>0}$, we define

$$L_s(r) := |\{\mathfrak{a} < v \leq n \mid N_s^+(v) = k \text{ and } N_{10s}^-(v) = 0 \text{ and } \max_{i \in E_s} X_{iv}^+ \Phi_{iv} \leq r\}|.$$

The random variable $L_s(r)$ has the following very useful property: assume that exactly s vertices are active at some time t , and let r be any positive real number; then at time $t + r$, there will be at least $\min\{\mathfrak{a} + L_s(r), 10s\}$ active vertices – indeed, unless $10s$ vertices are activated before time $t + r$, every vertex counted in $L_s(r)$ will be active by time $t + r$. (Here the value $10s$ has no deeper meaning: we just need some value sufficiently larger than s .) Therefore, if we want to show that many vertices turn active quickly, then we need to prove lower bounds for the variables $L_s(r)$. This is what we will do in the next lemma.

For the analysis, it turns out to be very useful to parametrize the number of active vertices at a given time as $s = x \cdot \mathfrak{a}_{\text{th}}/(1 - 1/k)$, for some $x > 0$. For this reason, we introduce the notation

$$\Lambda = \Lambda(n, p, k, \tau) := \frac{\mathfrak{a}_{\text{th}}}{1 - 1/k} = \left(\frac{(k-1)!}{(1-\tau)^k n p^k} \right)^{1/(k-1)}, \quad (2)$$

and note that Λ satisfies

$$\frac{(1-\tau)^k n p^k \Lambda^k}{(k-1)!} = \Lambda. \quad (3)$$

The following lemma shows essentially that, conditioned on the event that the values $|E_s|$ are very close to their expectations, it is unlikely that there is some $\mathfrak{a} \leq s \ll 1/p$ for which $L_s(r)$ is very small.

Lemma 7. *There exists a positive constant $c = c(\tau, k)$ such that if $\mathfrak{a} \geq \mathfrak{a}_{\text{th}}$ and $p \geq n^{-1}$, then the following holds for every $2k^2/((1-\tau)\mathfrak{a}) \leq \delta_0 \leq 1/(30k)$. Let $\eta \in [10k\delta_0, 1/2]$ and $\delta = \eta/(10k)$. Write \mathcal{E} for the event that $|E_s| \in (1 \pm \delta_0)(1 - \tau)s$ holds for all $s \geq \mathfrak{a}$. Then for every $\mathfrak{a} \leq s = x\Lambda \leq \min\{\delta/p, \delta/(\gamma p)\}$ and $r \in \mathbb{R}_{>0}$, we have*

$$\Pr[L_s(r) \geq (1 - \eta) \Pr[\Phi \leq r]^k x^k \Lambda / k \mid \mathcal{E}] \geq 1 - e^{-c\eta^2 \Pr[\Phi \leq r]^k x^k \Lambda}.$$

Proof. Write $\mathcal{E}_s(a)$ for the event that $|E_s| = a$, and $\mathcal{I}_s(b)$ for the event that $|I_s| = b$. Fix some $\mathfrak{a} \leq s \leq \min\{\delta/p, \delta/(\gamma p)\}$ and $r \in \mathbb{R}_{>0}$.

We first prove that for all integers $(1 - \tau)s/2 \leq a \leq s$ and $0 \leq b \leq 10s$ we have

$$\mathbb{E}[L_s(r) \mid \mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)] \geq (1 - \delta)^{13} \cdot \frac{na^k p^k}{k!} \cdot \Pr[\Phi \leq r]^k. \quad (4)$$

To see this, fix a vertex $\mathbf{a} < v \leq n$ arbitrarily and write $\mathcal{L}(v)$ for the event that $N_s^+(v) = k$ and $N_{10s}^-(v) = 0$ and $\max_{i \in E_s} X_{iv} \Phi_{iv} \leq r$. Using the conditional independence of $N_s^+(v)$ and $N_s^-(v)$ (see Remark 3), and the independence of the variables Φ_{iv} , we get

$$\begin{aligned} \Pr[\mathcal{L}(v) \mid \mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)] &= \Pr[N_s^+(v) = k \mid \mathcal{E}_s(a)] \cdot \Pr[N_{10s}^-(v) = 0 \mid \mathcal{I}_{10s}(b)] \cdot \Pr[\Phi \leq r]^k \\ &= \binom{a}{k} p^k (1 - p)^{a-k} \cdot (1 - \gamma p)^b \cdot \Pr[\Phi \leq r]^k \\ &\geq \left(1 - \frac{k^2}{a}\right) \frac{a^k p^k}{k!} (1 - p)^s \cdot (1 - \gamma p)^{10s} \cdot \Pr[\Phi \leq r]^k \\ &\geq (1 - \delta)^{12} \cdot \frac{a^k p^k}{k!} \cdot \Pr[\Phi \leq r]^k, \end{aligned}$$

using $s \leq \min\{\delta/p, \delta/(\gamma p)\}$, and the fact that $k^2/a \leq 2k^2/((1 - \tau)\mathbf{a}) \leq \delta_0 \leq \delta$.

By definition, we have $L_s(r) = |\{\mathbf{a} < v \leq n \mid \mathcal{L}(v)\}|$, and thus

$$\begin{aligned} \mathbb{E}[L_s(r) \mid \mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)] &\geq (n - \mathbf{a})(1 - \delta)^{12} \cdot \frac{a^k p^k}{k!} \cdot \Pr[\Phi \leq r]^k \\ &\geq (1 - \delta)^{13} \cdot \frac{na^k p^k}{k!} \cdot \Pr[\Phi \leq r]^k, \end{aligned}$$

for all a and b as above, proving (4). Here we used that $\mathbf{a} \leq \delta/p \leq \delta n$.

Now, observe that, by definition of the underlying probability space, the events $\{\mathcal{L}(v) \mid v \in [n]\}$ are conditionally independent given $\mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)$, for all choices of a and b . Then, by Lemma 4, writing $\mu_{a,b} := \mathbb{E}[L_s(r) \mid \mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)]$, we have

$$\Pr[L_s(r) < (1 - \delta)\mu_{a,b} \mid \mathcal{E}_s(a) \cap \mathcal{I}_{10s}(b)] < e^{-\delta^2 \mu_{a,b}/3} < e^{-\delta^2 \Pr[\Phi \leq r]^k (1 - \delta)^{13} na^k p^k / (3k!)}.$$

for all $(1 - \tau)s/2 \leq a \leq s$ and $0 \leq b \leq 10s$. If we condition on the event \mathcal{E} , then we may assume $a \in (1 \pm \delta)(1 - \tau)s$, and we get

$$\begin{aligned} \Pr[L_s(r) < (1 - \delta)^{14+k} \frac{\Pr[\Phi \leq r]^k (1 - \tau)^k n s^k p^k}{k!} \mid \mathcal{E}] \\ < e^{-\delta^2 \Pr[\Phi \leq r]^k (1 - \delta)^{14+k} (1 - \tau)^k n s^k p^k / (3k!)}. \end{aligned}$$

The lemma now follows using $s = x\Lambda$ with (3) and from $\delta = \eta/(10k)$, which implies that $1/2 \leq 1 - \eta \leq (1 - \delta)^{14+k}$ holds for $k \geq 2$. \square

Remark 8. For later reference, we just note here that (4) in the proof above, together with (3), implies that for every $\mathbf{a} \leq s = x\Lambda \leq \min\{1/(10kp), 1/(10k\gamma p)\}$ and $(1 - \tau)s/2 \leq a \leq s$, we have

$$\mathbb{E}[L_s(r) \mid |E_s| = a] = \Omega(x^k \Lambda \Pr[\Phi \leq r]^k).$$

Recall that, by the definition of $L_s(r)$, if there are exactly s active vertices at time t , then at time $t + r$ there are at least $\min\{\mathbf{a} + L_s(r), 10s\}$ active vertices. We now use this observation to obtain a lower bound on the growth of the process.

Corollary 9. *For every $\varepsilon > 0$, there exist positive constants $c_0 = c_0(\varepsilon, k, \Phi)$ and $\delta = \delta(\varepsilon, \gamma, k)$ such that for every function $f: \mathbb{N} \rightarrow \mathbb{N}$, the percolation process with $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$ and $n^{-1} \leq p \ll n^{-1/k}$ is a.a.s. such that at least $\min\{f(n)\mathbf{a}, \delta/p\}$ vertices are active at time $c_0 \log(f(n))$.*

Proof. Fix a sufficiently small constant $\eta = \eta(\varepsilon, k) \in (0, 1)$ and let $\delta := \eta/(10k)$. Note that $p \ll n^{-1/k}$ implies that $\mathbf{a} \geq \mathbf{a}_{\text{th}} = \omega(1)$, so that for large enough n , we have $\delta \geq 2k^2/((1 - \tau)\mathbf{a})$. Also, since $\mathbf{a} = \omega(1)$, by Lemma 5, we may assume that $|E_s| \in (1 \pm \delta)(1 - \tau)s$ holds for all $s \geq \mathbf{a}$.

Choose $r = r(k, \Phi)$ to be so large that $\Pr[\Phi \leq r]^k \geq 1 - \eta$. Applying Lemma 7, we get that for every $\mathbf{a} \leq s = x\Lambda \leq \min\{\delta/p, \delta/(\gamma p)\}$, we have

$$\Pr[L_s(r) \geq (1 - \eta)^2 x^k \Lambda / k] \geq 1 - e^{-\Omega(x^k \Lambda)} \geq 1 - e^{-cs},$$

for some positive constant $c = c(\eta, k, \tau)$. Since $\sum_{s \geq \mathbf{a}} e^{-cs} = e^{-c\mathbf{a}}/(1 - e^{-c}) = o(1)$ the process is such that a.a.s.

$$L_s(r) \geq (1 - \eta)^2 x^{k-1} s / k$$

holds for all $\mathbf{a} \leq s \leq \min\{\delta/p, \delta/(\gamma p)\}$. Thus

$$\frac{\mathbf{a} + L_s(r)}{s} \geq (1 - \eta)^2 \frac{\mathbf{a}/\Lambda + x^k/k}{x} \geq (1 - \eta)^2 (\mathbf{a}/\mathbf{a}_{\text{th}})^{\frac{k-1}{k}},$$

where the last inequality follows by minimizing over $x \geq \mathbf{a}/\Lambda = (1 - 1/k)(\mathbf{a}/\mathbf{a}_{\text{th}})$ and the minimum is obtained at $x^k = \mathbf{a}/\mathbf{a}_{\text{th}}$. As $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$, the definition of $L_s(r)$ implies that after every time period of constant length r , the number of active vertices is multiplied by a constant factor of size at least $\min\{(1 - \eta)^2(1 + \varepsilon)^{k-1}, 10\}$. If η is small enough, then $(1 - \eta)^2(1 + \varepsilon)^{k-1} > 1$, and the corollary follows. \square

Corollary 9 implies that the process grows at least at an exponential rate. In fact, it will turn out that in both the synchronous and the asynchronous case, the growth is actually much faster. However, the corollary already implies two useful facts. Firstly, regardless of the fraction τn of inhibitory vertices, the process will always reach $\Theta(1/p)$ active vertices. Secondly, in order to activate a constant multiple of the starting set we only need $O(1)$ time.

Corollary 10. *For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, \gamma, k) > 0$ such that for $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$ and $n^{-1} \leq p \ll n^{-1/k}$, the process a.a.s. activates at least δ/p vertices.*

Corollary 11. *For every $\varepsilon > 0$ and $c > 0$ there exist constants $T = T(\varepsilon, k, c, \Phi) > 0$ and $\delta = \delta(\varepsilon, \gamma, k) > 0$ such that if $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$ and $n^{-1} \leq p \ll n^{-1/k}$, then the percolation process a.a.s. activates at least $\min\{c\mathbf{a}, \delta/p\}$ vertices in time T .*

2.3 Phases of percolation

It is interesting to note that in the statements of Corollaries 10 and 11, the inhibition parameter τ is not mentioned at all. The reason for this is that, as long as there are $o(1/p)$ active vertices, the number of vertices that have even one active inhibitory neighbor is $o(n)$; in this sense, the behavior of the process is almost completely unaffected by the presence of inhibitory vertices until there are $\Omega(1/p)$ active vertices.

Thus, the evolution of the percolation process divides naturally into two separate phases: the *initial phase* $\mathbf{a} \leq s \ll 1/p$, during which the growth is largely unaffected by inhibition, and the *end phase* $s = \Omega(1/p)$, where many vertices start to have inhibitory neighbors.

If $\mathbf{a} = \Theta(\mathbf{a}_{\text{th}})$, then one can further subdivide the initial phase into two phases with $s = \Theta(\mathbf{a}_{\text{th}})$ and $\mathbf{a}_{\text{th}} \ll s \ll 1/p$, respectively. The former is called the *startup phase*, and Corollary 11 shows

that if $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}$, then the time spent in the startup phase is bounded from above by some constant. However, one can show that if $\mathfrak{a} = \Theta(\mathfrak{a}_{\text{th}})$, then this upper bound is close to the truth, i.e., the size of the active set really increases only by some (small) constant factor in each round. In contrast, once we have $s \gg \mathfrak{a}_{\text{th}}$, the rate of growth speeds up considerably. Thus we call this second phase the *explosion phase*. As we will see, the time that is spent in the explosion phase depends significantly on the distribution Φ of the signal delays: for the synchronous process (where Φ is identically one), the time spent in the explosion phase is $\log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(pn) + O(1)$, while for the asynchronous process (where Φ is exponentially distributed with mean one), it is $o(1)$.

3 Synchronous Bootstrap Percolation

In this section we study the synchronous bootstrap percolation process with inhibition. Recall that the synchronous process is defined by taking all edge delays to be constants $\Phi_{iv} = 1$. Then it is clear that for every vertex x_i , the time t_i at which x_i becomes active is either a non-negative integer or ∞ . For this reason, we can view the percolation process as happening in discrete rounds $t = 0, 1, 2, \dots$. We write

$$a_t := |\{x_i \mid i \in [n] \text{ and } t_i \leq t\}|$$

for the number of vertices that are active after round t , and

$$a^* := \max \{a_t \mid t \geq 0\}$$

for the number of vertices at termination.

For $\tau = 0$ (the $G_{n,p}$ case without inhibition), the process was analyzed in great detail in [12]. Among other results, it was shown that $\mathfrak{a}_{\text{th}}(n, p, k, 0)$ is the threshold for percolation in $G_{n,p}$, and moreover that the process with $\mathfrak{a} \leq (1 - \varepsilon)\mathfrak{a}_{\text{th}}$ will a.a.s. not even activate more than $k\mathfrak{a}/(k - 1)$ vertices. Moreover, the authors of [12] determined the typical number of rounds until percolation up to an additive constant.

In the case with inhibition, it is not clear that we percolate to a point where all (or at least most of) the excitatory vertices are active. Corollary 10 guarantees that inhibition essentially plays no role while we have at most δ/p active vertices, but from then on things may change. Our plan for the rest of this section is as follows. First we show that we can describe the dynamics of the percolation process very precisely up to δ/p active vertices. Then we show that this implies that the process with inhibition actually follows a complicated pattern, where the number of finally active vertices depends on the size on the starting set in a *non-monotone* way. We start by proving a concentration theorem.

Theorem 12. *For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, \gamma, k, \tau) > 0$ such that, for the sequence $(\hat{a}_t)_{t \geq 0}$ defined by*

$$\hat{a}_0 := \mathfrak{a} \quad \text{and} \quad \hat{a}_{t+1} := \hat{a}_t + (1 - \tau)^k n p^k \frac{\hat{a}_t^k}{k!}, \quad (5)$$

the synchronous process with $\mathfrak{a} \geq \max\{(1 + \varepsilon)\mathfrak{a}_{\text{th}}, (\log n)^{2+\varepsilon}\}$ and $p \gg n^{-1}$ a.a.s. satisfies

$$(1 - \varepsilon)\hat{a}_t \leq a_t \leq (1 + \varepsilon)\hat{a}_t$$

for all $t \geq 0$ such that $\hat{a}_t \leq \delta n$.

One can show that the requirement $\mathfrak{a} \geq (\log n)^{2+\varepsilon}$ is tight in the following sense: if we have $\mathfrak{a} < (\log n)^{2-\varepsilon}$ for some constant $\varepsilon > 0$, then with non-negligible probability the number of active vertices after the first round will deviate from its expectation by a factor that, accumulated over many rounds, makes it impossible for such a statement to hold. More precisely, assume that $\mathfrak{a} = (1 + \varepsilon)\mathfrak{a}_{\text{th}} \leq (\log n)^{2-\varepsilon}$, then the expectation of a_1 is

$$\mathbb{E}[a_1] \approx \mathfrak{a} + (1 - \tau)^k n p^k \mathfrak{a}^k / k! \stackrel{(3)}{=} \Theta(\mathfrak{a}).$$

Let $\delta = (\log n)^{\varepsilon/3-1} \ll \mathfrak{a}^{-1/2}$. By the tightness of the Chernoff bound (or by normal approximation), the probability that $a_1 > (1 + \delta)\mathbb{E}[a_1]$ is at least some constant. By the definition of the sequence \hat{a}_i , the factor $(1 + \delta)$ will blow up at a doubly exponential rate, and after i rounds, the uncertainty on a_i will be $(1 + \delta)^{\Theta(k^i)}$. We will see (cf. Lemma 15) that the number of rounds with $\hat{a}_t \leq \delta n$ is $\ell = \log_k \log(n) - O(1)$. So the uncertainty after ℓ rounds would be $(1 + (\log n)^{\varepsilon/2-1})^{\Theta(\log n)} \gg 1$, which shows that it is impossible for a_ℓ to be concentrated around \hat{a}_ℓ .

3.1 The speed of round-based percolation

In Subsection 2.2, we introduced a general approach for proving that the percolation progresses grows at least with a certain speed: if, at some point, there are s active vertices, then after waiting for a time period of length r , there will be at least $\min\{\mathfrak{a} + L_s(r), 10s\}$ active vertices. In the case of synchronous percolation, we can strengthen (and simplify) this statement a bit. Define, for every $s \in [n]$,

$$L_s := |\{\mathfrak{a} < v \leq n \mid N_s^+(v) = k \text{ and } N_s^-(v) = 0\}|.$$

Note that in comparison to the definition of $L_s(r)$, we replaced the condition $N_{10s}^-(v) = 0$ by $N_s^-(v) = 0$ and omitted the condition on the random variables Φ_{iv} . Nevertheless, due to the round-based nature of the synchronous process, we still can conclude: if there are $s = a_t$ active vertices at time t , then at time $t + 1$, there will be at least $\mathfrak{a} + L_s$ active vertices.

To prove concentration of the sequence $(a_t)_{t \geq 0}$, we need to show that this lower bound for a_{t+1} is more or less tight. To do this, we introduce a second set of random variables. For every $s \in [n]$, define

$$U_s := |\{\mathfrak{a} < v \leq n \mid N_s^+(v) \geq k\}|.$$

With this definition, it is clear that if at some time t , there are $s = a_t$ active vertices, then at time $t + 1$, there will not be more than $\mathfrak{a} + U_s$ active vertices. The next lemma says that for all $s \ll 1/p$, the upper and lower bounds U_s and L_s are not likely to differ by much.

Lemma 13. *There exists a positive constant $c = c(\tau, k)$ such that if $\mathfrak{a} \geq \mathfrak{a}_{\text{th}}$ and $p \geq n^{-1}$, then the following holds for every $2k^2/((1 - \tau)\mathfrak{a}) \leq \delta_0 \leq 1/(30k)$. Let $\eta \in [10k\delta_0, 1/2)$ and $\delta = \eta/(10k)$. Write \mathcal{E} for the event that $|E_s| \in (1 \pm \delta_0)(1 - \tau)s$ holds for all $s \geq \mathfrak{a}$. Then, for every $\mathfrak{a} \leq s = x\Lambda \leq \min\{\delta/p, \delta/(\gamma p)\}$, we have*

$$\Pr[L_s \geq (1 - \eta)x^k \Lambda / k \mid \mathcal{E}] \geq 1 - e^{-c\eta^2 x^k \Lambda}$$

and

$$\Pr[U_s \leq (1 + \eta)x^k \Lambda / k \mid \mathcal{E}] \geq 1 - e^{-c\eta^2 x^k \Lambda}.$$

Proof. Fix some $\mathfrak{a} \leq s = x\Lambda \leq \delta/p$. Since $L_s \geq L_s(1)$, the statement for L_s follows directly from Lemma 7. For the statement for U_s , given $0 \leq a \leq s$, write $\mathcal{E}_s(a)$ for the event that $|E_s| = a$. By Remark 3 we know that, conditioned on $\mathcal{E}_s(a)$, the variable U_s follows a binomial distribution. In order to obtain an upper bound on $\mu_a := \mathbb{E}[U_s \mid \mathcal{E}_s(a)]$ we use the following property of the binomial distribution: if $W \sim \text{Bin}(n, p)$ with $np \leq 1/2$, then we have (see for example [2])

$$\Pr[W \geq b] \leq (1 + 2np) \Pr[W = b] \quad \forall b \geq 0. \quad (6)$$

As $a \leq s \leq \delta/p$ this bound implies that

$$\mu_a = (n - \mathfrak{a}) \cdot \Pr[\text{Bin}(a, p) \geq k] \leq n \cdot (1 + 2pa) \cdot \frac{a^k p^k}{k!} \leq (1 + \delta)^2 \cdot \frac{na^k p^k}{k!}.$$

From here an application of the Chernoff bound (Lemma 4) gives that for every $(1 - \tau)s/2 \leq a \leq s$, we have

$$\Pr[U_s > (1 + \delta)^3 na^k p^k / k! \mid \mathcal{E}_s(a)] < e^{-\delta^2 \mu_a / 3} = e^{-\delta^2 \Omega(x^k \Lambda)},$$

since $L_s \leq U_s$ implies that $\mu_a \geq \mathbb{E}[L_s \mid \mathcal{E}_s(a)] = \Omega(x^k \Lambda)$, by Remark 8. Recall that the statement we want to prove conditions on the event \mathcal{E} , meaning that we can assume $a \in (1 \pm \delta)(1 - \tau)s$. The above bound thus implies

$$\Pr[U_s > (1 + \delta)^{3+k} n(1 - \tau)^k s^k p^k / k! \mid \mathcal{E}] < e^{-\delta^2 \Omega(x^k \Lambda)},$$

We have $(1 + \delta)^{k+3} = (1 + \eta/(10k))^{k+3} \leq 1 + \eta$, for all $\eta \in (0, 1)$ and $k \geq 1$. Then the lemma follows with an application of (3). \square

3.2 The expected trajectory $(\hat{a}_t)_{t \geq 0}$

Lemma 13 tells us that if there are $a_t = x\Lambda$ active vertices in round t , then in round $t + 1$, there will be

$$a_{t+1} \approx \mathfrak{a} + \frac{x^k \Lambda}{k} = \mathfrak{a} + (1 - \tau)^k n p^k \frac{a_t^k}{k!}$$

active vertices, using (3). This motivates the definition of a sequence $(\hat{a}_t)_{t \geq 0}$ in equation (5) in Theorem 12. Note that if we parametrize $\hat{a}_t = x\Lambda$, we get

$$\hat{a}_{t+1} = \mathfrak{a} + \frac{x^k}{k} \Lambda. \quad (7)$$

In the next lemma we establish a simple fact on the minimal growth of the sequence $(\hat{a}_t)_{t \geq 0}$.

Lemma 14. *For all $t \geq 0$, we have $\hat{a}_{t+1}/\hat{a}_t \geq (\mathfrak{a}/\mathfrak{a}_{\text{th}})^{\frac{k-1}{k}}$.*

Proof. Write $\hat{a}_t = x\Lambda$. Then we obtain from (7) that

$$\hat{a}_{t+1}/\hat{a}_t = \frac{\mathfrak{a}/\Lambda + x^k/k}{x}.$$

The minimum of this expression is achieved for $x = (\mathfrak{a}/\mathfrak{a}_{\text{th}})^{1/k}$, where its value is $(\mathfrak{a}/\mathfrak{a}_{\text{th}})^{\frac{k-1}{k}}$, completing the proof. \square

The bounds from the previous lemma are weak, but nevertheless best possible: the sequence (\hat{a}_t) grows very slowly at the beginning. Once, however \hat{a}_t is above, say, 2Λ , a doubly exponential growth kicks in, and implies that the total number of rounds of the process is just doubly logarithmic, as our next lemma shows.

Lemma 15. *For every $\varepsilon > 0$, there exists a constant $K = K(k, \varepsilon)$ such that for all large enough $n \in \mathbb{N}$, the following holds, provided $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}$ and $p = \omega(n^{-1})$:*

(i) $\hat{a}_t \geq n$ for all $t \geq \log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(pn) + K$, and

(ii) $\hat{a}_t \leq 1/p$ for all $t \leq \log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(pn) - K$.

Proof. First observe that by Lemma 14, there exists a constant $t_0 = t_0(\varepsilon, k)$ such that $\hat{a}_{t_0} \geq (\mathfrak{a}k/\mathfrak{a}_{\text{th}})\Lambda$. By (7), we see in particular that $\hat{a}_t = x\Lambda$ implies $\hat{a}_{t+1} \geq (x^k/k)\Lambda$. Using induction we get that for all $t \geq 0$, we have

$$\hat{a}_{t_0+t} \geq \frac{(\mathfrak{a}k/\mathfrak{a}_{\text{th}})^{k^t} \Lambda}{k^{1+k+k^2+\dots+k^{t-1}}} \geq \left(\frac{\mathfrak{a}k}{\mathfrak{a}_{\text{th}}k} \right)^{k^t} \Lambda = (\mathfrak{a}/\mathfrak{a}_{\text{th}})^{k^t} \Lambda.$$

It follows that for all $t \geq \log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(1/(p\Lambda))$, we have $\hat{a}_{t_0+t} \geq 1/p$, and so

$$\hat{a}_{t_0+t+1} \geq \mathfrak{a} + (1 - \tau)^k n/k! = \Omega(n),$$

whence $\hat{a}_{t_0+t+2} = \omega(n)$, using $p = \omega(n^{-1})$. Since $\log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(1/(p\Lambda))$ is within a constant difference of $\log_k \log_{(\mathfrak{a}/\mathfrak{a}_{\text{th}})}(pn)$, this proves (i).

For (ii), we may assume, again by Lemma 14, that there is some smallest constant $t_0 \geq 0$ such that $2\Lambda \leq \hat{a}_{t_0} \leq 1/p$. Now if $\hat{a}_t = x\Lambda \geq \mathfrak{a}$ for some $x \geq 2$, then, using (7) and $k \geq 2$, we have

$$\hat{a}_{t+1} = \mathfrak{a} + \frac{x^k \Lambda}{k} \leq (x + x^k/k)\Lambda \leq x^k \Lambda.$$

By induction, we thus have

$$\hat{a}_t \leq \hat{a}_{t_0+t} \leq (\hat{a}_{t_0}/\Lambda)^{k^t} \Lambda \leq (\hat{a}_{t_0}/\mathfrak{a}_{\text{th}})^{k^t} \Lambda$$

for all $t \geq 0$, and it follows that for all $t \leq \log_k \log_{(\hat{a}_{t_0}/\mathfrak{a}_{\text{th}})}(1/(p\Lambda))$, we have $\hat{a}_t \leq 1/p$. If $t_0 = 0$, then $\hat{a}_{t_0} = \mathfrak{a}$. If $t_0 > 0$, then $\hat{a}_{t_0}/\mathfrak{a}_{\text{th}} = O(1)$. In both cases, (ii) follows easily. \square

3.3 Initial phases – proof of Theorem 12

Assume that $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}$ holds for some constant $\varepsilon > 0$. We want to show that a.a.s.,

$$(1 - \varepsilon)\hat{a}_t \leq a_t \leq (1 + \varepsilon)\hat{a}_t$$

holds for all $t \geq 0$ such that $\hat{a}_t \leq \delta n$, where $\delta = \delta(\varepsilon, \gamma, k, \tau)$ is some positive constant. The idea is to proceed by induction over t . Recall that for $t = 0$ we have $\hat{a}_0 = a_0 = \mathfrak{a}$ by definition, so the base case is settled. The difficulty in the induction step is that from one round to the next the error bounds that we can prove will worsen. Therefore, instead of showing $a_t \in (1 \pm \varepsilon)\hat{a}_t$, we need to show $a_t \in (1 \pm \varepsilon_t)\hat{a}_t$ for an appropriate sequence $(\varepsilon_t)_{t \geq 0}$. Here is how we choose this sequence: set $\eta_0 = (\log n)^{-1-\varepsilon/3}$ and define η_t for $t \geq 1$ by

$$1 + \eta_t := (1 + \eta_0)(1 + 20k\hat{a}_t p \cdot \max\{1, \gamma\}) \leq 1 + \eta_0 + 40k\hat{a}_t p \cdot \max\{1, \gamma\}.$$

Finally, define the sequence $(\varepsilon_t)_{t \geq 0}$ recursively by

$$\varepsilon_0 := 0 \quad \text{and} \quad 1 + \varepsilon_t := (1 + \eta_{t-1}) \cdot (1 + \varepsilon_{t-1})^k = \prod_{i=0}^{t-1} (1 + \eta_i)^{k^{t-1-i}} \quad \text{for } t \geq 1,$$

where the last equality follows from a straightforward induction. Recall that we assume that $\mathfrak{a} \geq (\log n)^{2+\varepsilon}$, so that we have in particular that $\eta_0^2 \mathfrak{a} \geq (\log n)^{\varepsilon/3} = \omega(-\log \eta_0)$.

Lemma 16. *For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \gamma, k, \tau) > 0$ such that the following holds, assuming that $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{th}$ and $p = \omega(n^{-1})$. Define $(\varepsilon_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$ as above and write ℓ for the largest positive integer such that $\hat{a}_\ell \leq \delta n$. Then*

- (i) $\varepsilon_t \leq \varepsilon$ for all $0 \leq t \leq \ell$,
- (ii) $\ell = O(\log \log n)$, and
- (iii) $\eta_t \cdot \min\{1, \gamma^{-1}\} \geq (1 + \varepsilon_t)10k\hat{a}_t p$ holds for all $0 \leq t \leq \ell$.

We defer the technical proof of this lemma to the end of this subsection and first show how it can be used in order to complete the proof of Theorem 12.

Proof of Theorem 12. Assume the sequences $(\eta_t)_{t \geq 0}$ and $(\varepsilon_t)_{t \geq 0}$ are defined as above. As in the statement of Lemma 16, we define ℓ to be the largest positive integer t for which $\hat{a}_t \leq \delta n$, for some sufficiently small positive constant $\delta = \delta(\varepsilon, \gamma, k, \tau)$. By Lemma 16 (ii) and since $\eta_0^2 \mathfrak{a} \geq (\log n)^{\varepsilon/3}$, we know in particular that $\ell e^{-c\eta_0^2 \mathfrak{a}} = o(1)$ for any constant $c > 0$. For every $i \geq 0$, let $\delta_i = \eta_i/(10k)$. Write \mathcal{E} for the event that $|E_s| \in (1 \pm \delta_0)(1 - \tau)s$ holds for all $s \geq \mathfrak{a}$.

Let $x_i := \hat{a}_i/\Lambda$ and let $s_i^{(1)} := (1 - \varepsilon_i)\hat{a}_i$ and $s_i^{(2)} := (1 + \varepsilon_i)\hat{a}_i$. Observe that by (7) and Lemma 14, we have

$$x_i^k \Lambda/k = \hat{a}_{i+1} - \mathfrak{a} \geq \varepsilon \mathfrak{a}.$$

Note that Lemma 16 (iii) implies that $s_i^{(1)}, s_i^{(2)} \leq \min\{\delta_i/p, \delta_i/(\gamma p)\}$. One easily checks that the other conditions of Lemma 13 are met, so we obtain that there is a constant $c > 0$ such that

$$\Pr[L_{s_i^{(1)}} \geq (1 - \eta_i)(1 - \varepsilon_i)^k(\hat{a}_{i+1} - \mathfrak{a}) \mid \mathcal{E}] \geq 1 - e^{-c\eta_0 \mathfrak{a}}$$

and

$$\Pr[U_{s_i^{(2)}} \leq (1 + \eta_i)(1 + \varepsilon_i)^k(\hat{a}_{i+1} - \mathfrak{a}) \mid \mathcal{E}] \geq 1 - e^{-c\eta_0 \mathfrak{a}}.$$

We have $(1 + \eta_i)(1 + \varepsilon_i)^k = 1 + \varepsilon_{i+1}$ and one can see that this implies that $(1 - \eta_i)(1 - \varepsilon_i)^k \geq 1 - \varepsilon_{i+1}$. Moreover, by Lemma 5 and the fact that $\delta_0^2 \mathfrak{a} = \omega(-\log \delta_0)$, we have $\Pr[\mathcal{E}] = 1 - o(1)$. Thus, by the union bound, with probability $1 - 2\ell e^{-c\eta_0 \mathfrak{a}} = 1 - o(1)$, we have

$$L_{s_i^{(1)}} \geq (1 - \varepsilon_{i+1})(\hat{a}_{i+1} - \mathfrak{a}) \quad \text{and} \quad U_{s_i^{(2)}} \leq (1 + \varepsilon_{i+1})(\hat{a}_{i+1} - \mathfrak{a}) \quad (8)$$

for all $0 \leq i < \ell$. In the following, we assume that this is the case.

We now prove by induction that for each $0 \leq i \leq \ell$, we have

$$(1 - \varepsilon_i)\hat{a}_i \leq a_i \leq (1 + \varepsilon_i)\hat{a}_i. \quad (9)$$

Note that by Lemma 16 (i), this will complete the proof. Since $a_0 = \hat{a}_0$, Equation (9) holds trivially for $i = 0$. For the induction, assume that it holds for a given $i \geq 0$, that is, assume $s_i^{(1)} \leq a_i \leq s_i^{(2)}$. Then by the definition of the sets $L_{s_i^{(1)}}$ and $U_{s_i^{(2)}}$, we have

$$\mathfrak{a} + L_{s_i^{(1)}} \leq a_{i+1} \leq \mathfrak{a} + U_{s_i^{(2)}}.$$

By (8), this implies

$$(1 - \varepsilon_{i+1})\hat{a}_{i+1} \leq a_{i+1} \leq (1 + \varepsilon_{i+1})\hat{a}_{i+1},$$

completing the proof. \square

Proof of Lemma 16. By Lemma 15, we know that $\ell \leq \log_k \log_{(\mathbf{a}/\mathbf{a}_{\text{th}})}(np) + K$, for some constant $K = K(k, \varepsilon)$, so (ii) is immediate.

To prove (i), fix some $0 \leq t \leq \ell$. Using the fact that $\log(1 + x) \leq x$ holds for all $x > -1$, we get

$$\log(1 + \varepsilon_t) = \sum_{i=0}^{t-1} k^{t-1-i} \log(1 + \eta_i) \leq \eta_0 \sum_{i=0}^{t-1} k^{t-1-i} + \max\{1, \gamma\} \cdot 40kp \sum_{i=0}^{t-1} k^{t-1-i} \hat{a}_i.$$

We bound the two terms individually. Since $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$, we have

$$\eta_0 \sum_{i=0}^{t-1} k^{t-1-i} = \eta_0 \frac{k^t - 1}{k - 1} \leq \eta_0 \frac{k^\ell - 1}{k - 1} \leq \frac{k^{K(k, \varepsilon)} \log_{(\mathbf{a}/\mathbf{a}_{\text{th}})}(np) - 1}{(\log n)^{1+\varepsilon/3} (k - 1)} = o(1).$$

Now consider the smallest integer $t_0 \geq 0$ such that $\hat{a}_{t_0} > 4k^3\Lambda$. By Lemma 14, we know that t_0 is bounded. Thus, using the upper bound on t , (2) and $pn = \omega(1)$ we obtain

$$\max\{1, \gamma\} \cdot 40kp \sum_{i=0}^{t_0-1} k^{t-1-i} \hat{a}_i = \Theta(p\Lambda k^t) = O(\log(pn)(pn)^{-1/(k-1)}) = o(1)$$

and it thus remains to bound the quantity $\max\{1, \gamma\} \cdot 40kp \sum_{i=t_0}^{t-1} k^{t-1-i} \hat{a}_i$.

By (7) we have $\hat{a}_{i+1}/\hat{a}_i \geq (4k^3)^{k-1}/k \geq 2k$ for every $i \geq t_0$. By induction, it follows that for every $t_0 \leq i < t - 1$, we have $\hat{a}_{t-1} \geq (2k)^{t-1-i} \hat{a}_i$. Moreover, by the definition (5) of \hat{a}_ℓ

$$\delta n \geq \hat{a}_\ell = \mathbf{a} + (1 - \tau)^k n p^k \frac{\hat{a}_{\ell-1}^k}{k!},$$

implying that for large enough n

$$\hat{a}_{t-1} \leq \hat{a}_{\ell-1} \leq \left(k! \frac{\delta n - \mathbf{a}}{(1 - \tau)^k n p^k} \right)^{1/k} \leq \frac{\delta^{1/k} k}{(1 - \tau)p}.$$

We get

$$40kp \sum_{i=t_0}^{t-1} k^{t-1-i} \hat{a}_i \leq 40kp \sum_{i=t_0}^{t-1} k^{t-1-i} (2k)^{-(t-1-i)} \hat{a}_{t-1} \leq 40kp \hat{a}_{t-1} \leq 40k^2 \delta^{1/k} / (1 - \tau).$$

Therefore, if δ is small enough, then $\log(1 + \varepsilon_t) \leq \log(1 + \varepsilon)$, and so $\varepsilon_t \leq \varepsilon$, which proves (i).

By (i), we have $\varepsilon_t \leq \varepsilon < 1$, and so

$$\eta_t \cdot \min\{1, \gamma^{-1}\} \geq 20k \hat{a}_t p > (1 + \varepsilon_t) 10k \hat{a}_t p,$$

proving (iii). \square

3.4 End phase – proof of Theorem 1

In this subsection, we will study the effect of the inhibition parameter τ on the number of active vertices at termination. Theorem 12 shows in particular that the process does not stop while $a_t = o(1/p)$ (since $a_t = o(1/p)$ implies $a_{t+1} = o(n)$), and the growth of the process during that time does not depend in any significant way on the number of inhibitory vertices. The situation changes during the very last rounds.

Lemma 17. *For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, \gamma, k, \tau) > 0$ such that the synchronous bootstrap percolation process satisfies the following, assuming $\max\{(1 + \varepsilon)\mathbf{a}_{th}, (\log n)^{2+\varepsilon}\} \leq \mathbf{a} \leq \delta/p$ and $p \gg n^{-1}$. Let ℓ denote the the largest positive integer such that $\hat{a}_\ell \leq \delta n$.*

- (i) *If $\tau < 1/(1 + \gamma)$ then a.a.s. the process almost percolates in at most $\ell + 2$ rounds. If moreover $p \gg \log n/n$, then the process completely percolates in at most $\ell + 2$ rounds.*
- (ii) *If $\tau > 1/(1 + \gamma)$ and $p \gg \log n/n$, then there exists some constant $C = C(\tau, \gamma) > 0$ such that if $\hat{a}_\ell \geq C(\log n)/p$, then a.a.s. the process stops with $(1 - \varepsilon)\hat{a}_\ell \leq a^* \leq (1 + \varepsilon)\hat{a}_\ell$.*
- (iii) *If $\tau > 1/(1 + \gamma)$ and $p \gg \log n/n$, then for every $\alpha > 0$, there exists a constant $C' > 0$ such that if $C'/p \leq \hat{a}_\ell \leq \alpha n/(1 + \varepsilon)$, then a.a.s. the process stops with $a^* \leq \alpha n$.*

Some remarks are in order. By Lemma 15, we already know that ℓ is, up to an additive constant, at most $\log_k \log_{(\mathbf{a}/\mathbf{a}_{th})}(pn)$. Then (i) shows that the number of rounds to percolation a.a.s. takes one of only two possible (deterministic) values $\ell + 1$ and $\ell + 2$. If $\hat{a}_\ell > C(\log n)/p$, then the proof actually implies that a.a.s. the process percolates in exactly $\ell + 1$ rounds.

Lemma 17 spares out the border cases (a) $\tau = 1/(1 + \gamma)$, and (b) $\tau > 1/(1 + \gamma)$ and $\hat{a}_\ell \leq C'/p$. We also do not determine the size of the final active set for the regime $\hat{a}_\ell \leq C(\log n)/p$. These regimes show a slightly richer, but also more complicated behavior. Here even a harmless factor of $1 + o(1)$ in the size of the starting set can shift² the size of the ℓ -th set from $\Theta(1/p)$ to $\omega((\log n)/p)$, so every effect that depends on the property $C'/p \leq \hat{a}_\ell \leq C(\log n)/p$ should be considered unstable.

Proof of Lemma 17. From Theorem 12 we know that we can choose $\delta > 0$ such that

$$(1 - \varepsilon)\hat{a}_t \leq a_t \leq (1 + \varepsilon)\hat{a}_t$$

holds for all $0 \leq t \leq \ell$, where ℓ is as in the statement of the theorem. Also, by Lemma 5, we may assume that

$$|E_s| \in (1 \pm \delta_0)(1 - \tau)s \quad \text{and} \quad |I_s| \in (1 \pm \delta_0)\tau s$$

holds for all $s \geq \mathbf{a}$, for some $\delta_0 = \delta_0(n) = o(1)$. Using the definition of the sequence $(\hat{a}_t)_{t \geq 0}$ and since, by definition of ℓ , we have $\hat{a}_{\ell+1} > \delta n$, we can easily check that if n is large enough, then we have $a_\ell \geq \delta^{1/k}/p$. We will prove the three statements of Lemma 17 separately.

First consider (i), that is, assume that $\tau < 1/(1 + \gamma)$. In a first step we show by a case distinction that $a_{\ell+1} = \Theta(n)$. Let $s := a_\ell$ and let $C \in \mathbb{N}$ be a large enough constant (that we define below). Assume first that $sp \geq C$. Let $\xi = \xi(\tau, \gamma) > 0$ be so small that $(1 - \xi)^2(1 - \tau) \geq (1 + \xi)^2\tau\gamma$; such a choice is possible since $\tau < 1/(1 + \gamma)$. Then the assumption that

$$|E_s| \in (1 \pm \delta_0)(1 - \tau)s \quad \text{and} \quad |I_s| \in (1 \pm \delta_0)\tau s$$

²We do not give a formal proof of this fact, it follows essentially from the calculations in the proof of Theorem 1 below.

implies in particular that $(1 - \xi)|E_s|p \geq (1 + \xi)|I_s|\gamma p$. By the Chernoff bounds (Lemma 4), we have

$$\Pr[N_s^-(v) > (1 + \xi/2)|I_s|\gamma p] \leq e^{-\frac{1}{12}\xi^2|I_s|\gamma p} \leq e^{-\frac{1}{12}\xi^2(1-\delta_0)\tau\gamma sp}$$

and

$$\Pr[N_s^+(v) < (1 - \xi)|E_s|p] \leq e^{-\frac{1}{3}\xi^2|E_s|p} \leq e^{-\frac{1}{3}\xi^2(1-\delta_0)(1-\tau)sp}.$$

Since $sp \geq C$, by choosing C large enough, we can assume that both these probabilities are at most $1/3$. Then with probability at least $1/3$, we have

$$N_s^+(v) \geq (1 - \xi)|E_s|p \geq (1 + \xi)|I_s|\gamma p \geq k + (1 + \xi/2)|I_s|\gamma p \geq k + N_s^-(v),$$

where we used that for large enough C , we have $\xi|I_s|\gamma p/2 \geq k$. For the second case assume now that $\delta^{1/k} \leq sp < C$, which implies in particular that $p < C/s = o(1)$. In this case, the probability that $N_s^-(v) = 0$ is at least

$$(1 - \gamma p)^{|I_s|} \geq (1 - \gamma p)^s \geq (1 - \gamma p)^{C/p} \geq 2^{-2C\gamma},$$

and the probability that $N_s^+(v) \geq k$ is at least

$$\binom{|E_s|}{k} p^k (1 - p)^k \geq \frac{|E_s|^k}{2k^k} p^k \geq \frac{p^k s^k (1 - \tau)^k}{4k^k} \geq \frac{\delta(1 - \tau)^k}{4k^k}.$$

So both probabilities are bounded from below by positive constants. Since $N_s^+(v)$ and $N_s^-(v)$ are conditionally independent, we see that with positive probability, we have $N_s^+(v) \geq k + N_s^-(v)$. Summarizing, we proved in both cases that for every vertex v , $N_s^+(v) \geq k + N_s^-(v)$ occurs independently with some nonzero constant probability. Another application of Chernoff thus implies that after round $\ell + 1$, a.a.s. a linear fraction of all vertices is active. Since $p = \omega(n^{-1})$, we then have $\Pr[N_{a_{\ell+1}}^+(v) \geq k + N_{a_{\ell+1}}^-(v)] = 1 - o(1)$ for all vertices $v \in [n]$. This implies that in round $\ell + 2$, there are $n - o(n)$ active vertices. If we assume additionally that $p = \omega(\log n/n)$, then using the Chernoff bounds and the union bound, we actually obtain that a.a.s. we have $N_{a_{\ell+1}}^+(v) \geq k + N_{a_{\ell+1}}^-(v)$ for all $v \in [n]$, which proves that all vertices are active in round $\ell + 2$, showing (i).

To show (ii), assume $\tau > 1/(1 + \gamma)$ and that $\hat{a}_\ell \geq C(\log n)/p$ holds for some large constant $C = C(\tau, \gamma) > 0$ (chosen below), so that $s := a_\ell \geq (1 - \varepsilon)C(\log n)/p$. To prove that the process stops with s active vertices, it is enough to show that every vertex $v \in [n]$ is such that $N_s^-(v) \geq N_s^+(v)$. Fix any vertex $v \in [n]$ and choose a constant $\xi = \xi(\tau, \gamma) > 0$ so small that

$$(1 + \xi)|E_s|p \leq (1 + \xi)(1 + \delta_0)(1 - \tau)sp \leq (1 - \xi)(1 - \delta_0)\tau\gamma sp \leq (1 - \xi)|I_s|\gamma p;$$

such a choice is possible since $\tau > 1/(1 + \gamma)$. If C is sufficiently large, then by the Chernoff bounds (Lemma 4) we get

$$\Pr[N_s^-(v) < (1 - \xi)|I_s|\gamma p] \leq e^{-\xi^2(1-\delta_0)\tau\gamma sp/3} \ll n^{-1},$$

and

$$\Pr[N_s^+(v) > (1 + \xi)|E_s|p] \leq e^{-\xi^2(1-\delta_0)\tau sp/3} \ll n^{-1}.$$

So by the union bound, a.a.s. every vertex $v \in [n]$ satisfies $N_s^-(v) \geq N_s^+(v)$ and the process will stop.

Finally, for (iii), let $\tau > 1/(1+\gamma)$ and suppose that we are given some $\alpha > 0$. Let $C' = C'(\alpha)$ be large enough and assume that

$$(1 - \varepsilon)C'/p \leq (1 - \varepsilon)\hat{a}_\ell \leq a_\ell \leq (1 + \varepsilon)\hat{a}_\ell \leq \alpha n.$$

If $(1 + \varepsilon)\hat{a}_\ell \geq \alpha n/2$, then by (ii), the process will stop with $a^* = a_\ell$ active vertices, so assume from now on that $a_\ell \leq (1 + \varepsilon)\hat{a}_\ell < \alpha n/2$.

First, if there is no $t \geq \ell$ such that $a_t \geq \alpha n/2$, then we are done. Otherwise, let t_0 be the smallest $t \geq \ell$ with this property. The same arguments as in (ii) show that the process stops with $a^* = a_{t_0}$. Thus, it suffices to show that $a_{t_0} \leq \alpha n$. To prove this, it is enough to show that with probability tending to one, we have $a_{t+1} \leq a_t + \alpha n/2$ for all $t \geq \ell$. To see this recall that $a_t \geq a_\ell \geq (1 - \varepsilon)C'/p$. Thus, if we choose C' large enough, then we have $\Pr[N_s^+(v) \geq N_s^-(v)] \leq \alpha/4$ for every vertex $v \in [n]$ and for every $s \geq a_t$, using $\tau > 1/(1+\gamma)$. Then, by the Chernoff bound, the probability that $a_{t+1} - a_t \geq \alpha n/2$ is $o(n^{-1})$. Since there can be at most n rounds in total until the process stops (there are only n vertices), the union bound easily shows that a.a.s., $a_{t+1} \leq a_t + \alpha n/2$ holds for all $t \geq \ell$, completing the proof. \square

Proof of Theorem 1. Observe that Corollary 11, Lemma 17 allows us to restrict ourselves to the case $\tau > 1/(1+\gamma)$. Given any real number \hat{a}_0 , we can define a sequence $(\hat{a}_t)_{t \geq 0}$ by (5), as in the statement of Theorem 12. Our first goal is to show that this sequence is sufficiently robust against rounding down the starting value \hat{a}_0 .

For this, fix any $C_2 > C_1 > 0$, and assume that $C_1 \mathfrak{a}_{\text{th}} \leq \hat{a}_0 \leq C_2 \mathfrak{a}_{\text{th}}$ is any real number. Denote by ℓ the largest positive integer t for which $\hat{a}_t \leq n$. From Lemma 15 we know that

$$\ell = \log_k \log_{(\hat{a}_0/\mathfrak{a}_{\text{th}})}(pn) + O(1) = \log_k \log(pn) + O(1).$$

Let $(\hat{b}_t)_{t \geq 0}$ denote the sequence defined by the same recursion as \hat{a}_t , but with an initial value of $\lfloor \hat{a}_0 \rfloor$, i.e., $\hat{b}_0 = \lfloor \hat{a}_0 \rfloor$ and $\hat{b}_{t+1} = \hat{b}_t + (1 - \tau)^k n p^k \hat{b}_t^k / k!$. We will show by induction that for all $t \geq 0$, we have $\hat{b}_t / \hat{a}_t \geq (1 - 1/\hat{a}_0)^{k^t}$. For $t = 0$ this immediately follows from $\hat{b}_0 \geq \hat{a}_0 - 1$. For the inductive step assume $\hat{b}_{t-1} / \hat{a}_{t-1} \geq (1 - 1/\hat{a}_0)^{k^{t-1}}$. Using Equation (7) on page 13, we have

$$\frac{\hat{b}_t}{\hat{a}_t} = \frac{\hat{b}_0 + (\hat{b}_{t-1}/\Lambda)^k \Lambda/k}{\hat{a}_0 + (\hat{a}_{t-1}/\Lambda)^k \Lambda/k} \geq \frac{\left(1 - \frac{1}{\hat{a}_0}\right) \hat{a}_0 + \left(1 - \frac{1}{\hat{a}_0}\right)^{k^t} (\hat{a}_{t-1}/\Lambda)^k \Lambda/k}{\hat{a}_0 + (\hat{a}_{t-1}/\Lambda)^k \Lambda/k} \geq \left(1 - \frac{1}{\hat{a}_0}\right)^{k^t},$$

as claimed. Thus the error in \hat{a}_ℓ caused by rounding \hat{a}_0 down to the next integer satisfies

$$1 \geq \frac{\hat{b}_\ell}{\hat{a}_\ell} \geq \left(1 - \frac{1}{\hat{a}_0}\right)^{k^\ell} \geq \left(1 - \frac{1}{C_1(\log n)^{2+\varepsilon}}\right)^{\Theta(\log(pn))} \rightarrow 1,$$

by the assumption that $\hat{a}_0 \geq C_1 \mathfrak{a}_{\text{th}} \geq C_1(\log n)^{2+\varepsilon}$. This means that for the asymptotic size of \hat{a}_ℓ , it does not matter whether \hat{a}_0 is rounded down to the next smallest integer or not.

To complete the proof of Theorem 1, we will show that for every constant C_1 , there exists a constant C_2 such that for every function $\log n/p \ll f(n) \ll n$, there exists a function $C_1 \leq c(n) \leq C_2$ such that a.a.s., the process with $\mathfrak{a} = \lfloor c(n)\Lambda \rfloor$ stops with $(1+o(1))f(n)$ active vertices. Observe that it suffices to consider constants C_1 that are sufficiently large so that the inequalities below hold.

Consider the process with $\mathfrak{a} = C_1 \Lambda$. Recall that we assume that $\mathfrak{a}_{\text{th}} \geq (\log n)^{2+\varepsilon}$ holds for some constant $\varepsilon > 0$. Since we may assume that $C_1 \geq 1 + \varepsilon$, Theorem 12 implies that there exists some $\delta > 0$ such that a.a.s.,

$$(1 - \varepsilon)\hat{a}_t \leq a_t \leq (1 + \varepsilon)\hat{a}_t$$

holds for all $0 \leq t \leq \ell$, where ℓ is the largest integer such that $\hat{a}_\ell \leq \delta n$. Define ℓ_0 to be the largest integer such that $\hat{a}_{\ell_0} \leq f(n)/(1 + \varepsilon)$, and note that, since $f(n) \ll n$, we have $\ell_0 \leq \ell$ for all large enough n . Thus we have $a_{\ell_0} \leq (1 + \varepsilon)\hat{a}_{\ell_0} \leq f(n)$ a.a.s..

Observe also that for large enough n , we have $f(n) \leq \hat{a}_{\ell_0}np$, which is obvious if $\hat{a}_{\ell_0} \geq 1/p$ and otherwise follows from

$$f(n)/(1 + \varepsilon) \leq \hat{a}_{\ell_0+1} = \hat{a}_0 + (1 - \tau)^k \hat{a}_{\ell_0}^k \frac{np^k}{k!} \leq \hat{a}_0 + np\hat{a}_{\ell_0}/k! \leq np\hat{a}_{\ell_0}/(1 + \varepsilon).$$

We will show that if one multiplies \hat{a}_0 with a large enough constant factor c_0 , then \hat{a}_{ℓ_0} increases by a factor of $\omega(pn)$. This will imply, by the intermediate value theorem, that there exists some $c = c(n) \in [C_1, c_0C_1]$ such that a starting value $\hat{a}_0 = c\Lambda$ results in $\hat{a}_{\ell_0} = f(n)$. Then, by the argument above, and by Lemma 17 (ii) (using $f(n) \gg (\log n)/p$), the process with $\mathfrak{a} = \lfloor c\Lambda \rfloor$ will stop after ℓ_0 rounds with $(1 + o(1))f(n)$ active vertices. Since C_1 is an arbitrary constant and since $\Lambda = \Theta(\mathfrak{a}_{\text{th}})$, this will complete the proof of the theorem.

So consider a sequence $(\hat{b}_t)_{t \geq 0}$ defined by $\hat{b}_0 = c_0C_1\Lambda$ and by the same recursion (5), with \hat{a} replaced by \hat{b} . Our goal is to show that $\hat{b}_{\ell_0}/\hat{a}_{\ell_0} = \omega(pn)$. Write $\hat{b}_t = c_t\hat{a}_t$ and $\hat{a}_t = x_t\Lambda$. Using (7) and the fact that x_t is monotonically increasing, we see that for all $t \geq 0$, we have

$$c_{t+1} = \frac{\hat{b}_{t+1}}{\hat{a}_{t+1}} = \frac{c_0C_1 + (c_t x_t)^k/k}{C_1 + x_t^k/k} \geq c_t^k \left(1 - \frac{C_1}{C_1 + x_t^k/k}\right) \geq c_t^k \left(1 - \frac{C_1}{C_1 + C_1^k/k}\right).$$

In particular, if C_1 and c_0 are large enough, then we have $c_1 \geq 2c_0$ and $c_t \geq c_{t-1}^k/2$ for all $t > 0$. By induction it follows that for all $t > 0$, we have

$$c_t \geq \frac{c_1^{k^{t-1}}}{2^{(k^{t-1}-1)/(k-1)}} \geq c_0^{k^{t-1}}.$$

Since $f(n) \geq 1/p$, Lemma 15 tells us that $\ell_0 \geq \log_k \log(pn) - O(1)$, where the constant in the $O(1)$ term does not depend on c_0 . Therefore, if c_0 is large enough, we get

$$\frac{\hat{b}_{\ell_0}}{\hat{a}_{\ell_0}} = c_{\ell} \geq c_0^{\Omega(\log pn)} = \omega(pn),$$

completing the proof. □

4 Asynchronous Bootstrap Percolation

In the second part of the paper we consider the bootstrap percolation process with an additional temporal component. More precisely, we assume that all edges have independent delays distributed according to $\text{Exp}(1)$. Recall that these transmission delays correspond to the random variables Φ_{iv} in the probability space introduced in Section 2.1.

The main difference of this model to the synchronous case studied in the previous section is that the activation no longer takes place in rounds, but that vertices turn active at individual times. Recall that we write t_s for the time at which the s -th vertex turns active. Note that we may assume without loss of generality that no two vertices become active at the same time (except for the vertices in the starting set).

4.1 Initial phases

The goal of this subsection is to describe the behavior of the process in the range where few vertices are active. In this range, inhibition does not play an important role.

Lemma 18. *For every $\varepsilon > 0$, there exists a constant $T = T(\varepsilon, k) > 0$ such that the asynchronous process with $\mathfrak{a} \geq (1 + \varepsilon)\mathfrak{a}_{th}$ and $n^{-1} \ll p \ll n^{-1/k}$ satisfies the following. For every constant $C > 0$, a.a.s.,*

- (i) $t_{C/p} \leq T$,
- (ii) $t_{C/p} - t_{1000\Lambda} \leq 1 + o(1)$, and
- (iii) $t_{C/p} - t_s = o(1)$ for all $s = \omega(\mathfrak{a}_{th})$.

Proof. Let $\delta_0 \in (0, 1/2)$ be a sufficiently small constant. By Lemma 5, and since $p \ll n^{-1/k}$ implies $\mathfrak{a} \geq \mathfrak{a}_{th} = \omega(1)$, we can condition the process on the event \mathcal{E} that

$$|E_s| \in (1 \pm \delta_0)(1 - \tau)s$$

holds for all $s \geq \mathfrak{a}$. Moreover, by Lemma 7 (with $\eta = 1/2$), we know that for every $\mathfrak{a} \leq s = x\Lambda \leq 1/(20kp)$ and $r > 0$, we have

$$\Pr[L_s(r) < \Pr[\Phi \leq r]^k x^k \Lambda / (2k) \mid \mathcal{E}] < e^{-c \Pr[\Phi \leq r]^k x^k \Lambda}, \quad (10)$$

for some positive constant $c = c(\tau, k)$ that is independent of r and x .

From Corollary 11, we know that a.a.s. the process reaches at least $s_0 = \min\{1000\Lambda, \delta/p\}$ active vertices after some constant time $T_0 = T_0(\varepsilon, k)$, for an appropriate $\delta = \delta(\varepsilon, \gamma, k) > 0$. Starting from T_0 , we examine successive time intervals of lengths $1/2, 1/4, 1/8 \dots$, respectively, and compute the number of active excitatory vertices after each interval. So, for every $i \geq 1$, define $T_i = T_0 + \sum_{j=1}^i 2^{-j}$ and let s_i be the number of vertices active at time T_i . We claim that a.a.s. we have

$$s_i \geq \min\{10^{i+3}\Lambda, \delta/p\} \quad (11)$$

for every $i \geq 0$. Write \mathcal{S}_i for the event that (11) holds for i . Since the lengths of our time intervals sum up to 1, the occurrence of $\bigcap_{i \geq 0} \mathcal{S}_i$ implies that at time $T_0 + 1$ there are at least δ/p active vertices. We will show by induction that

$$\Pr[\mathcal{S}_i \mid \mathcal{E}] \geq 1 - \sum_{j=1}^i e^{-c 2^j \Lambda} \quad (12)$$

holds for all $i \geq 0$. The case $i = 0$ is evident by the choice of s_0 , so let us assume that (12) holds for some $i \geq 0$. Let $\Delta_i := T_{i+1} - T_i = 2^{-(i+1)}$.

Write \mathcal{A} for the event that $s_i \geq \delta/p$. Since $s_i \geq \delta/p$ implies $s_{i+1} \geq \delta/p$, we have

$$\Pr[\mathcal{S}_{i+1} \mid \mathcal{S}_i \cap \mathcal{E} \cap \mathcal{A}] = 1.$$

On the other hand, if \mathcal{A} does not occur, then \mathcal{S}_i implies that $s_i \geq 10^{i+3}\Lambda$, and in this case, we have $s_{i+1} \geq \min\{\mathfrak{a} + L_{10^{i+3}\Lambda}(\Delta_i), 10^{i+4}\Lambda\}$, by the definition of $L_{10^{i+3}\Lambda}(\Delta_i)$. By (10), we have

$$\Pr[L_{10^{i+3}\Lambda}(\Delta_i) < \Pr[\Phi \leq \Delta_i]^k 10^{k(i+3)}\Lambda / (2k) \mid \mathcal{E}] < e^{-c \Pr[\Phi \leq \Delta_i]^k 10^{k(i+3)}\Lambda}$$

for a suitable constant $c > 0$. Using the fact that $\Phi \sim \text{Exp}(1)$, that $k \geq 2$, and that $e^{-x} \leq 1 - x/2$ for $0 < x \leq 1/2$ we get

$$\Pr[\Phi \leq \Delta_i]^k 10^{k(i+3)} = \left(1 - e^{-2^{-(i+1)}}\right)^k 10^{k(i+3)} \geq 2^{-k(i+2)} \cdot 10^{k(i+3)} \geq 2k10^{i+4}$$

for all $i \geq 0$. Then we get

$$\Pr[\mathcal{S}_{i+1} \mid \mathcal{S}_i \cap \mathcal{E} \cap \overline{\mathcal{A}}] \geq 1 - e^{-c2k10^{i+4}\Lambda} \geq 1 - e^{-c2^{i+1}\Lambda}.$$

Therefore,

$$\Pr[\mathcal{S}_{i+1} \mid \mathcal{E}] \geq \Pr[\mathcal{S}_i \mid \mathcal{E}] \left(\Pr[\mathcal{A}] + \Pr[\overline{\mathcal{A}}](1 - e^{-c2^{i+1}\Lambda}) \right) \geq 1 - \sum_{j=1}^{i+1} e^{-c2^j\Lambda},$$

using the induction hypothesis. This completes the proof of (12).

Finally, since $\sum_{j=1}^{\infty} e^{-c2^j\Lambda} = o(1)$, it follows by the union bound that $\bigcap_{i \geq 0} \mathcal{S}_i$ occurs with probability $1 - o(1)$, i.e., that a.a.s. at least δ/p vertices are active at time $T_0 + 1$. Since T_0 is the first time at which $\min\{1000\Lambda, \delta/p\}$ vertices are active, this also shows that the time to go from 1000Λ to δ/p active vertices is at most 1. Additionally, the same proof shows that a.a.s. for every $i \geq 0$, the time from $s_i = 10^{i+2}\Lambda$ to δ/p active vertices is at most $T_0 + 1 - T_i$, which implies that a.a.s., the time to go from $\omega(\mathbf{a}_{\text{th}})$ to δ/p active vertices is $o(1)$.

Let T' denote the earliest time at which δ/p vertices are active. To complete the proof of (i-iii), it suffices to show that for all constants $\rho > 0$ and $C > 0$, a.a.s. at least C/p vertices are active at time $T' + \rho$. Note that for a fixed vertex $v \in [n]$, the events $N_{C/p}^- = 0$ and $N_{\delta/p}^+ = k$ and $\max_{i \in E_{\delta/p}} \{X_{iv}\Phi_{iv} \leq \rho\}$ occur simultaneously with some positive constant probability. Thus, by the Chernoff bounds (Lemma 4), we get that a.a.s. a constant fraction of all vertices, say cn vertices for some $c = c(k, \gamma, \rho, C)$ satisfies these three conditions. Hence, by time $T' + \rho$, at least $\min\{cn, C/p\}$ vertices are active. This proves the claim since $C/p \leq cn$ for large enough n . \square

4.2 End phase – proof of Theorem 2

Write $a(t)$ for the number of vertices that are active at time t and write

$$S^+(v, t) := |\{0 \leq i \leq a(t) \mid X_{iv}^+ = 1 \text{ and } \Psi_i = +1 \text{ and } t_i + \Phi_{iv} \leq t\}|$$

and

$$S^-(v, t) := |\{0 \leq i \leq a(t) \mid X_{iv}^- = 1 \text{ and } \Psi_i = -1 \text{ and } t_i + \Phi_{iv}^- \leq t\}|$$

for the number of excitatory/inhibitory signals that have reached vertex v at time t . For brevity, we will also write $S(v, t) := S^+(v, t) + S^-(v, t)$.

Observe that a vertex is part of the final active set exactly if there is some time $t \geq 0$ such that $S^+(v, t) - S^-(v, t) \geq k$. Thus, our main goal will be to describe the evolution of the random variables $S^+(v, t) - S^-(v, t)$, for different vertices v .

We start by proving some properties that are satisfied by most vertices during the end phase of the process.

Lemma 19. *For every $\varepsilon > 0$, the asynchronous process with $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{\text{th}}$ and $n^{-1} \ll p \ll n^{-1/k}$ satisfies the following. For all constants $\xi \in (0, 1/3)$ and $C_0 > 0$, and for every sufficiently large constant $C > 0$, if $n \in \mathbb{N}$ is large enough, then with probability at least $1 - \xi$, all but at most ξn vertices $v \in [n]$ satisfy:*

- (i) $N_s^+(v) \in (1 \pm \xi)(1 - \tau)sp$ and $N_s^-(v) \in (1 \pm \xi)\tau\gamma sp$ for all $s \geq C/p$,
- (ii) $S(v, t_{C/p}) = 0$, and
- (iii) $S(v, t_{1/p} + \xi) \geq C_0$.

Proof. Fix some vertex $v \in [n]$. We will show that for C chosen sufficiently large, the vertex v satisfies (i)–(iii) with probability at least $1 - \xi^2$. Then it will follow from Markov's inequality that the probability that more than ξn vertices fail to satisfy (i)–(iii) is at most ξ .

We first show (i). Recall that by Lemma 5 we may assume that with probability $1 - \xi^2/8$ we have

$$|E_s| \in (1 \pm \xi/8)(1 - \tau)s \quad \text{for all } s \geq (1 + \varepsilon)\mathfrak{a}_{\text{th}}, \quad (13)$$

where, as before, E_s denotes the set of excitatory vertices among the first s active vertices. We know that $N_s^+(v)$ is binomially distributed with parameters $|E_s|$ and p . That is, given (13) an application of Chernoff bounds (Lemma 4) gives us

$$\Pr[N_s^+(v) \notin (1 \pm \xi/2)(1 - \tau)sp] \leq e^{-c\xi^2(1-\tau)sp},$$

for some constant c that does not depend on ξ . By the union bound we deduce that the claim holds for all $s = (C + i)/p$ with $i \in \mathbb{N}_0$, as $\sum_{i \geq 0} e^{-c\xi^2(1-\tau)(C+i)} = O(e^{-c\xi^2(1-\tau)C})$ and the term on the right hand side can be made smaller than, say, $\xi^2/8$ by choosing $C = C(\xi, \tau)$ sufficiently large. Now consider an $s = (C + i + r)/p$ for some $r \in (0, 1)$ and let $\bar{s} = (C + i + 1)/p$. Then

$$N_s^+(v) \leq N_{\bar{s}}^+(v) \leq (1 + \xi/2)(1 - \tau)\bar{s}p \leq (1 + \xi)(1 - \tau)sp,$$

whenever $C = C(\xi, \tau)$ is sufficiently large. The lower bound for $N_s^+(v)$ follows similarly. The statement for $N_s^-(v)$ follows similarly, with $|E_s|$ replaced by $|I_s|$ and p replaced by γp . This then shows that (i) holds for v with probability at least $1 - \xi^2/4$.

Next we show (ii). The statement is trivial if $\mathfrak{a} \geq C/p$, so assume otherwise. Assume further for the time being that $N_{C/p}(v) \leq (1 + \xi)(1 - \tau + \tau\gamma)C$, i.e., that v has only constantly many active neighbors at time $t_{C/p}$. By (i), v satisfies this condition with probability at least $1 - \xi^2/4$.

Choose $\delta = \delta(\xi) > 0$ so that $N_{\delta/p}(v) = 0$ holds with probability at least $1 - \xi^2/5$. Since $\delta/p = \omega(\mathfrak{a}_{\text{th}})$ (this is implied by $p = \omega(n^{-1})$), Lemma 18 tells us that $t_{C/p} - t_{\delta/p} = o(1)$ a.a.s.. The probability that one of the (constantly many) active neighbors of v at time $t_{C/p}$ has already sent its signal to v is thus $o(1)$. Therefore, the probability that v satisfies both (i) and (ii) is at least $1 - \xi^2/2$ for sufficiently large n .

Lastly, let us prove (iii). Assume that v satisfies both (i) and (ii). By the previous point, this happens with probability at least $1 - \xi^2/2$. In particular, assume that $N_{C/p}^+(v) \geq (1 - \xi)(1 - \tau)C$. If C is large enough, then with probability at least $1 - \xi^2/4$, at least C_0 excitatory neighbors of v will send their signal to v before time $t_{C/p} + \xi/2$, i.e., $S(v, t_{C/p} + \xi/2) \geq C_0$. However, by Lemma 18, we have $t_{C/p} + \xi/2 \leq t_{1/p} + \xi$, which shows that v satisfies (i)–(iii) with probability at least $1 - \xi^2$. As noted above, an application of Markov's inequality completes the proof. \square

Before we come to the technical part of the proof, we give an intuition for the result. Let us consider a typical vertex v at time $t_{C/p}$. The previous lemma shows that, although v has many active neighbors at time $t_{C/p}$, none of their signals has arrived at vertex v at that time. Moreover, we can assume that throughout the process, roughly the correct fraction of the neighbors of v are excitatory. I.e., when s vertices are active there are about $(1 - \tau)sp$ excitatory neighbors and $\tau\gamma sp$ inhibitory ones. Recall that we assumed that the delays (the variables Φ_{iv}) are exponentially

distributed, that is, they are *memoryless*, which means that every neighbor whose signal has not yet arrived is *equally likely* to be the next to deliver its signal. Therefore, we would expect that, as the signals come in, the difference $S^+(v, t) - S^-(v, t)$ performs a random walk with a bias close to $\frac{1-\tau}{1-\tau+\gamma}$, in which case the probability that the vertex v becomes active is just the probability that such a random walk ever reaches k . This is the idea for the rest of the analysis. We will need the following basic facts about random walks.

Lemma 20. *Let X_1, X_2, \dots, X_n be a sequence of independent random variables, each of which is equal to 1 with probability $\beta \in [0, 1]$ and -1 otherwise. Consider the biased random walk $Z_i = X_1 + X_2 + \dots + X_i$. Then there exists for every $\varepsilon > 0$ and $k \in \mathbb{N}$ a constant $C_0 = C_0(\varepsilon, k)$ such that the following is true:*

$$\Pr[\forall n \geq C_0 : Z_n \in (2\beta - 1)n \pm \varepsilon n] \geq 1 - \varepsilon$$

and

$$\Pr[\exists i \leq C_0 \text{ s.t. } Z_i = k] \in (1 \pm \varepsilon) \cdot \min \left\{ 1, \frac{\beta^k}{(1 - \beta)^k} \right\}.$$

Proof. The first fact follows immediately from Chernoff bounds and implies the second fact whenever $\beta > 1/2$. For $\beta \leq 1/2$ the second fact follows from [8], Problem 5.3.1. \square

For every vertex $v \in [n]$ and every $i \in \mathbb{N}$, define $X_i^{(v)}$ to be 1 if the i -th signal arriving in v is excitatory, and -1 otherwise. Here we assume that in the asynchronous process, no two signals arrive simultaneously, which is the case with probability 1. Then we can define $Z_i^{(v)} := X_1^{(v)} + X_2^{(v)} + \dots + X_i^{(v)}$, and we know that the vertex v becomes active with the arrival of the first signal that causes $Z_i^{(v)}$ to become k , if such a signal exists. As outlined before, we will show that $Z_i^{(v)}$ follows (essentially) a random walk with bias

$$\beta := \frac{1 - \tau}{1 - \tau + \gamma\tau}. \quad (14)$$

If $\tau \geq 1/(1 + \gamma)$, then $\beta \leq 1/2$, and by Lemma 20 we would expect that roughly $n\beta^k/(1 - \beta)^k$ vertices are activated. There are two problems which complicate the analysis: the first being that the processes $(Z_i^{(v)})_{i \in \mathbb{N}}$ and $(Z_i^{(u)})_{i \in \mathbb{N}}$ are not independent for different vertices u and v , and the second being that for a fixed vertex v , the variables $X_i^{(v)}$ and $X_j^{(v)}$ are not independent for $i \neq j$, meaning that $(Z_i^{(v)})_{i \in \mathbb{N}}$ is not a true random walk. However, the following lemma tells us that at least for the first C_0 incoming signals these problems do not matter.

Lemma 21. *For every $0 < \varepsilon, \zeta < 1/3$, there exists some $T = T(\varepsilon, k) > 0$ independent of ζ such that for every large enough constant $C_0 > 0$, the asynchronous percolation process with $\mathbf{a} \geq (1 + \varepsilon)\mathbf{a}_{th}$ and $n^{-1} \ll p \ll n^{-1/k}$ satisfies the following: for every large enough $n \in \mathbb{N}$, with probability at least $1 - \zeta$,*

- *if $\tau > 1/(1 + \gamma)$, then $a(T) \geq \beta^k n / (1 - \beta)^k - \zeta n$ and at most $\beta^k n / (1 - \beta)^k + \zeta n$ vertices get activated by their first C_0 incoming signals. Moreover, all but at most ζn vertices v are such that $Z_{C_0}^{(v)} \leq -(1 - \zeta)(1 - 2\beta)C_0$.*
- *if $\tau \leq 1/(1 + \gamma)$, then $a(T) \geq n - \zeta n$.*

Proof. Assume first that $\tau > 1/(1 + \gamma)$ and note that this is equivalent to $0 < \beta < 1/2$, where β is as in (14). Let $\xi = \xi(\zeta) > 0$ be sufficiently small (to be fixed below) and choose $\eta = \eta(\zeta) > 0$ such that $0 < \beta - \eta \leq \beta + \eta \leq 1/2$ and

$$\left(\frac{\beta}{1-\beta}\right)^k - \frac{\zeta}{2} \leq \left(\frac{\beta-\eta}{1-(\beta-\eta)}\right)^k - \eta \leq \left(\frac{\beta+\eta}{1-(\beta+\eta)}\right)^k + \eta \leq \left(\frac{\beta}{1-\beta}\right)^k + \frac{\zeta}{2}. \quad (15)$$

Moreover, we may assume that $\eta < \zeta(1 - 2\beta)/3$, which implies $(2(\beta + \eta) - 1)C_0 + \eta C_0 \leq (1 - \zeta)(2\beta - 1)C_0$ (note that $2\beta - 1$ is negative). We will apply Lemma 20 for $\beta - \eta$ and for $\beta + \eta$. Clearly, whenever $C_0 = C_0(\eta) = C_0(\zeta)$ is sufficiently large, then for both values the probability in Lemma 20 is within η of the limit if we only consider the first C_0 terms. We will also apply Lemma 20 with $\beta + \eta$, and we can assume that C_0 is so large that the probability that $Z_{C_0} \in (2\beta - 1)C_0 \pm \eta C_0$ is at least $1 - \xi$.

Let $C = C(\xi, C_0) = C(\zeta)$ be so large that $C_0 \leq \xi(1 - \xi)(1 - \tau)C$ and that Lemma 19 guarantees that with probability $1 - \xi$ there exists a set $V_0 \subseteq [n]$ of size at least $|V_0| \geq (1 - \xi)n$ such that for all $v \in V_0$ we have

- (i) $N_s^+(v) \in (1 \pm \xi)(1 - \tau)sp$ and $N_s^-(v) \in (1 \pm \xi)\tau\gamma sp$ for all $s \geq C/p$,
- (ii) $S(v, t_{C/p}) = 0$, and
- (iii) $S(v, t_{1/p} + \xi) \geq C_0$.

We will prove that for every vertex $v \in V_0$ and every time $t \geq 0$ such that $S(v, t) < C_0$, the first signal arriving in v after time t is excitatory with probability within $\beta \pm \eta$. Moreover, we will show: (\star) these bounds hold regardless of the states of all other vertices.

Before proving this claim we show that this suffices to prove the first bullet point. Applying Lemma 20 with respect to $\beta - \eta$ and $\beta + \eta$, together with our assumptions on C_0 and Lemma 19 we observe that the probability that a vertex $v \in V_0$ gets active by receiving the first C_0 signals is within $(\beta \pm \eta)^k / (1 - \beta \pm \eta)^k \pm \eta = \beta^k / (1 - \beta)^k \pm \zeta/2$ by (15). By applying Chernoff bounds (which we may, because of (\star)), and since we may choose $\xi = \xi(\zeta)$ small enough, this then implies that a.a.s. at least $(1 - \xi)(\beta^k / (1 - \beta)^k - \zeta/2)|V_0| \geq \beta^k / (1 - \beta)^k n - \zeta n$ vertices become active with one of the first C_0 incoming signals. (Note that the error that we get from Chernoff is in the order $e^{-\Theta(n)}$, which is smaller than ξ for all large enough n .) Similarly, a.a.s. at most $(1 + \xi)(\beta^k / (1 - \beta)^k + \zeta/2)|V_0| + (n - |V_0|) \leq \beta^k / (1 - \beta)^k + \zeta n/2$ vertices become active by one of the first C_0 signals. In addition, Lemma 20 (with $\beta + \eta$) and Chernoff bounds show that for at least $(1 - \xi)|V_0| \geq n - \zeta n$ vertices, we have $Z_{C_0}^{(v)} \leq (1 - \zeta)(2(\beta + \eta) - 1)C_0 + \eta C_0 \leq (1 - \zeta)(2\beta - 1)C_0$. Since $t_{1/p} + \xi < t_{1/p} + 1$ can be upper bounded by a constant T by Lemma 18, this implies the claim of the first bullet point.

So consider some $v \in V_0$ and $t \geq 0$ with $S(v, t) < C_0$. Assume that the first signal that arrives at v after time t does so at time t^* . Let $s \in [n]$ be such that $t^* \in (t_s, t_{s+1}]$. By (ii), we know that we have $s + 1 > C/p$, as no signals arrive before time $t_{C/p}$. In the following, condition the process on the value of s . By the memorylessness of the exponential distribution, the conditional probability that the new signal is excitatory is

$$\frac{N_s^+(v) - S^+(v, t)}{N_s(v) - S(v, t)}.$$

By our choice of C , we deduce that (i) implies that

$$S^+(v, t) \leq S(v, t) < C_0 \leq \xi N_{C/p}(v) < \xi N_{s+1}(v) \leq \xi(N_s(v) + 1).$$

Also, again by (i), we have

$$\frac{N_s^+(v)}{N_s(v)} = \frac{N_s^+(v)}{N_s^+(v) + N_s^-(v)} \in (1 \pm 3\xi)\beta$$

for all $s \geq C/p$. Therefore, if ξ is small enough, the probability of an excitatory signal is at least

$$\frac{(1 - 3\xi)\beta N_s(v) - \xi(N_s(v) + 1)}{N_s(v)} \geq \beta - \eta$$

and at most

$$\frac{(1 + 3\xi)\beta N_s(v)}{N_s(v) - \xi(N_s(v) + 1)} \leq \beta + \eta.$$

Note that these bounds hold independently of the value of s .

Now consider the case $\tau \geq 1/(1 + \gamma)$, which is equivalent to $\beta \geq 1/2$. For every $\eta > 0$, and every large enough C_0 (depending on η), by a similar argument as above, with probability $1 - \xi$, there exist a set V_0 of $(1 - \xi)n$ vertices such that for every vertex $v \in V_0$, each of the first C_0 signals arriving in v is excitatory with probability at least $1/2 - \eta$, and moreover, $S(v, t_{1/p} + \xi) \geq C_0$. Then, if C_0 is large enough, Lemma 20 and the Chernoff bound show that a.s. at least $(1 - \xi) \left(\frac{1/2 - \eta}{1/2 + \eta} \right)^k |V_0|$ vertices of V_0 become active with one of the first C_0 signals, and for sufficiently small ξ and η this number is at least $n - \zeta n$. Since for the vertices of V_0 , we have $S(v, t_{1/p} + \xi) \geq C_0$, and since $t_{1/p} + \xi$ can be upper bounded by a constant T (by Lemma 18), this shows that $a(T) \geq n - \zeta n$. \square

It remains to prove that if $\tau \geq 1/(1 + \gamma)$, then not too many vertices become active. This is the content of the following lemma.

Lemma 22. *For every $\varepsilon > 0$, the asynchronous percolation process with $\tau \geq 1/(1 + \gamma)$, $(1 + \varepsilon)\mathbf{a}_{th} \leq \mathbf{a} \ll n$, and $n^{-1} \ll p \ll n^{-1/k}$ a.s. does not activate more than $n\beta^k/(1 - \beta)^k + o(n)$ vertices, where β is given by (14).*

Proof. If $\tau = 1/(1 + \gamma)$, the statement is trivial since then $\beta = 1/2$. So assume that $\tau > 1/(1 + \gamma)$ and thus $\beta < 1/2$.

Let $\zeta \in (0, 1)$ be arbitrarily small, but fixed. Let $\xi = \xi(\zeta) > 0$ be sufficiently small (to be fixed below), and let $C_0 = C_0(\zeta) > 0$ be sufficiently large (so that we can apply Lemma 21 and the inequalities below hold). Lastly, assume that n is sufficiently large. Let V_0 be a set of $n - \zeta n$ vertices such that

$$N_s^+(v) \leq (1 + \xi)\beta N_s(v) \tag{16}$$

holds for every $v \in V_0$ and s such that $S(v, t_s) > 0$. If n is large enough, then such a set exists with probability at least $1 - \zeta$, by Lemma 19 (i) and (ii).

Recall that by Lemma 21, for every $\zeta > 0$, for sufficiently large n with probability at least $1 - \zeta$ at most $\beta^k n / (1 - \beta)^k + \zeta n$ vertices get activated by their first C_0 incoming signals. Thus, we only need to show that there are few vertices that get activated later than by the first C_0 signals. More precisely, we will show that for large enough n , with probability at least $1 - 3\zeta$ there are at most $3\zeta n$ vertices v for which there is $i > C_0$ such that $Z_i^{(v)} = k$.

Again by Lemma 21 with probability $1 - \zeta$ there is a set $U_0 \subseteq V_0$ of size $|U_0| \geq (1 - 2\zeta)n$ such that $Z_{C_0}^{(v)} \leq (1 - \zeta)(2\beta - 1)C_0$ for all $v \in U_0$, for sufficiently large n . The proof will be complete if we show that for every $v \in U_0$, the probability that $Z_i^{(v)} = k$ holds for some $i > C_0$ is at most ζ^2 (the statement then follows from Markov's inequality and the requirement that only $\mathbf{a} \ll n$ vertices are active initially). Given some $v \in V_0$, write $\mathcal{A}_{i,i^*}^{(v)}$ for the event that

- (i) $Z_i^{(v)} \leq (1 - \zeta)(2\beta - 1)i$,
- (ii) $Z_{i^*}^{(v)} = k$, and
- (iii) $Z_j^{(v)} > (1 - \zeta)(2\beta - 1)j$ for all $i < j < i^*$.

Then it suffices to show that $\Pr[\bigcup_{i^* > i \geq C_0} \mathcal{A}_{i,i^*}^{(v)}] \leq \zeta^2$. We first show that for every $v \in U_0$ and $i \geq C_0$, we have

$$\Pr\left[\bigcup_{i^* > i} \mathcal{A}_{i,i^*}^{(v)}\right] \leq \left(\frac{\beta + \xi}{1 - \beta - \xi}\right)^{(1-\zeta)(1-2\beta)i+k}.$$

For this, assume that $j > C_0$ is such that $Z_j^{(v)} > (1 - \zeta)(2\beta - 1)j$. Let t be any time at which v has seen exactly $j - 1$ signals and assume that the j -th signal arrives in v at time $t^* \in (t_s, t_{s+1}]$. In the following, condition on the value of s . By (16), we know that $N_s^+(v) \leq (1 + \xi)\beta N_s(v)$. Moreover, from $Z_j^{(v)} = 2S^+(v, t^*) - j$ and our assumption on $Z_j^{(v)}$ we get

$$S^+(v, t) \geq S^+(v, t^*) - 1 \geq \beta j - \zeta(2\beta - 1)j/2 - 1 \geq (1 + \xi)\beta S(v, t),$$

where for the last inequality, we used the facts that $S(v, t) = j - 1$, that $2\beta - 1$ is negative, that $j > C_0$, where C_0 is sufficiently large, and the fact that we may choose $\xi = \xi(\zeta)$ to be sufficiently small. Then we have

$$\Pr[Z_{j+1}^{(v)} - Z_j^{(v)} > 0] = \frac{N_s^+(v) - S^+(v, t)}{N_s(v) - S(v, t)} \leq \frac{(1 + \xi)\beta(N_s(v) - S(v, t))}{N_s(v) - S(v, t)} \leq \beta + \xi.$$

By Lemma 20, this means that for every $i \geq C_0$ such that (i) holds, the probability that there exists some i^* for which (ii) and (iii) hold is at most

$$\left(\frac{\beta + \xi}{1 - \beta - \xi}\right)^{(1-\zeta)(1-2\beta)i+k},$$

where, since $\beta < 1/2$, we can assume that $(\beta + \xi)/(1 - \beta - \xi) < 1$. For large enough C_0 , the union bound yields

$$\Pr\left[\bigcup_{i^* > i \geq C_0} \mathcal{A}_{i,i^*}\right] \leq \sum_{i \geq C_0} \left(\frac{\beta + \xi}{1 - \beta - \xi}\right)^{(1-\zeta)(1-2\beta)i} \leq \zeta^2,$$

completing the proof. \square

Proof of Theorem 2. The statement for $\tau \geq 1/(1 + \gamma)$ is an immediate consequence of Lemmas 21 and 22.

For the case $\tau < 1/(1 + \gamma)$, we know by Lemma 21 that if $T = T(\varepsilon)$ is sufficiently large, then $s = n - o(n)$ vertices are active at time T , which takes care of the first part of this case.

For the case $\tau < 1/(1 + \gamma)$ and $p = \omega(\log n/n)$, the Chernoff and union bounds show that a.a.s. every vertex $v \in [n]$ is such that $N_s^+(v) = (1 + o(1))(1 - \tau)pn$ and $N_n^-(v) = (1 + o(1))\tau\gamma pn$. Using again the Chernoff and union bounds, within a time period of constant length depending on $\tau < 1/(1 + \gamma)$, every vertex receives at least $|N_n^-(v)| + k$ excitatory signals and becomes active. \square

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